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Optimal dynamic clearing for interbank payments

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Abstract

We investigate the optimal clearing policy for a financial system composed of a number of member banks and a central bank in a dynamic setting, when new debts between member banks are generated over time. The central bank clears the debts among members in the system in order to minimize the costs, including the set-up cost of each clearing, the coordination cost of clearing the net debts and the liquidity cost of uncleared debts. We formulate the problem using dynamic programming via state space reduction that provides a tractable framework to analyze and compute the optimal policy. We characterize the structure of the optimal policy and show that it is optimal for the central bank either to clear all the debts in the system or not to clear at all in each period under mild cost structure. This structure leads to efficient computation of the optimal clearing policy. We further characterize the optimal clearing frequency based upon the deterministic approximation for the debt process. We show that this approximation is close to optimal in broad settings and demonstrate that is practical for industrial-size problems using data from Payments Canada.

1 Introduction

With new technologies and digital commerce driving the need for safer and faster payments, there is a trend among payments clearing companies to launch real-time payment systems among institution members. For example, the Clearing House Payments Company (formerly known as the New York Clearing House Association) launched the Real Time Payments system in November 2017 to provide real-time clearing and settling of payments for the U.S. financial institutions. As another example, Payments Canada has been designing two real-time payment systems, the Lynx High-value Payments System and the Real-time Rail, through a modernization program since 2015.

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However, such payment systems typically require participants to hold large amounts of intraday liquidity because payments are settled on a gross basis (Byck & Heijmans, 2021). Moreover, without sophisticated coordination and synchronization of payment flows, these systems may cause significant settlement delays and payment congestion that impair their efficiency (Afonso & Shin, 2011; Bartolini et al., 2010). To address these problems, liquidity saving mechanisms (LSMs) have been proposed and widely adopted as a way for the real-time gross settlement (RTGS) systems to facilitate early settlement of payments and improve their efficiency (Byck & Heijmans, 2021; Jurgilas & Martin, 2013). An LSM is a centralized mechanism that matches possible offsetting payment obligations to minimize liquidity requirements for settling a payment network at different time points. The most notable example is the Clearing House Interbank Payments System (CHIPS), which has a patented algorithm to match and net payment obligations in the clearing process and provides the most efficient liquidity saving mechanism available today.

Based on current LSM practices, We consider a salient feature of payment systems in optimizing the clearing policy: the payment obligations are created between members constantly over time, and usually in both directions. For example, the participants in the Mexican RTGS system - SPEI, can reuse incoming payments as a source of intraday liquidity based on the pattern of the payment dynamics (Alexandrova-Kabadjova et al., 2015). An LSM that myopically clears all debts at each moment may not be optimal from a dynamic perspective. To illustrate, consider a payment system with an intermediate level of debts (we use payment obligation and debt interchangeably hereafter). From a static point of view, the LSM should match and clear all debts immediately. But the clearing cost may be reduced if some debts are left in the system to buffer future payment obligations in the opposite direction. As another example, if the central decision maker can predict a large amount of payment obligations building up in the system in the short term, then it may be optimal for the LSM to take preemptive actions to ease the potential shocks. Such a dynamic approach would require the central decision maker to have a reasonable forecasting model for future events, which has become more and more common in RTGS systems, especially when the liquidity flows in the system exhibit predictable daily patterns. For example, the intraday payment patterns in the European RTGS system, TARGET2, demonstrate fairly strong consistency and can greatly help understand the underlying liquidity needs of the participants in the system (Craig et al., 2018).

In this paper, we propose an LSM-type central design in a payment system that matches payment obligations between banks while balancing the payment flows intertemporally. We characterize the optimal *amount* and *timing* to clear debts. Here by clearing debts we broadly refer to a range of activities of the central decision maker including arranging, coordinating and overseeing the settlement (the actual transfer of funds) of payments among banks to control the debt level in the payment system. In our model, at any period, given the current information of the system, the decision maker, i.e., the payments settlement organization, faces a set of trade-offs: the clearing cost versus the liquidity cost of uncleared debts (i.e., the system inefficiency caused by settlement delays), the (certain) cost of proactively taking action now versus the uncertainty from the shocks in the future. We formulate, analyze and characterize the optimal clearing policy for the central decision maker in this dynamic setting. We note that the practice in high-value payment systems, where payments are automatically cleared once risk tests are passed, may not be minimizing these costs.

The contributions of this paper are three-fold. First, this is the first theoretical study that takes a fully dynamic perspective on the clearing of payment systems and proposes a model that internalizes the intertemporal costs in a unified framework. Previous studies on interbank payments largely focused on two or three-period models. Our formulation relies on dynamic programming, a tool that has been widely used in operations research and computer science and allows us to derive the optimal clearing policy and provide a computational approach for the implementation of the algorithm in practice. This framework has not been explored in the literature and may bring new insights to the clearing mechanisms: the efficiency of the system can be improved if the central decision maker has better control of the clearing time, which is typically not the case in practice. Second, we establish that our approach is data efficient. A common challenge faced by a central decision maker is that the coordination of institutions usually requires the members to report their balance sheets in detail, or the performance of the centralized approach would be significantly undermined. By a novel application of the technique *state space reduction*, we show that to obtain the optimal clearing policy, the centralized decision maker only needs to keep track of the aggregated net debt of each bank. This fact greatly enhances the applicability of our study. Third, we reveal a deep insight into the structure of the optimal clearing policy. We show that the optimal clearing policy preserves a simple *clearing-all* structure: for certain cost structures identified in the paper, it is optimal for the central decision maker to make binary choices in each period, either clearing all the debts in the system or not clearing at all. Even when this cost structure does not hold, the optimal clearing policy has a succinct structure. The result allows us to develop computationally feasible algorithms to support the decision making in practice. In particular, we show that when the debt is approximated by a stationary deterministic process, the asymptotically optimal clearing cycle length can be derived in a closed form (Section 3.3). Such a closed form derivation wouldn't be possible without the clearing-all structure. Using an extensive numerical study on synthetic and real data, we demonstrate that this approximation policy would work remarkably well in practice and is very efficient to compute.

The paper is organized as follows. We first provide a brief literature review. In Section 2, we describe the model setup. In Section 3, we study the optimal clearing policies in a dynamic setting. We prove that the dynamic clearing model of payment networks can be greatly simplified using state space reduction and derive structures for the optimal clearing policy. In Section 3.3, we analyze special cases of the dynamic clearing problem to illustrate the power of our approach. In Section 4, we present numerical experiments to illustrate the various performance measures of the optimal policy and several benchmarks. We also test our method on a simple dataset from Payments Canada. Section 5 concludes the paper.

1.1 Literature Review

Early studies of clearing interbank payments focused on the real-time gross settlement (RTGS) and the deferred net settlement (DNS) mechanisms in terms of liquidity needs and systemic risk control (see, e.g., Freixas & Parigi, 1998; Kahn et al., 2003; Kahn & Roberds, 1998; Rochet & Tirole, 1996). While the RTGS mechanism suggests a policy of clearing all payment obligations immediately and requires extensive liquidity, the DNS mechanism suggests a policy of clearing all debts only at the end of the day and allows offsetting of debts to economize on the use of liquidity. Subsequent studies try to incorporate the liquidity saving advantage of DNS into RTGS and analyze the design of the liquidity saving mechanism (LSM) (see, e.g., Byck & Heijmans, 2021; Jurgilas & Martin, 2013; Martin & McAndrews, 2008, 2010). Most studies have been focusing on a two or three-period model, with the exception of Castro et al. (2021), Galbiati and Soramäki (2011), Kahn and Roberds (1998), and Koeppl et al. (2008, 2012). Castro et al. (2021) and Galbiati and Soramäki (2011) respectively use reinforcement learning techniques and agent-based model to simulate equilibrium behaviour of banks in multi-period settings. Kahn and Roberds (1998) use standard Brownian motion to model the payment flow between banks and analyze the incentive problems in interbank payment networks. Koeppl et al. (2008, 2012) differentiate the transaction stages from the settlement stages and analyze the role of payment systems and clearing houses for buyers, sellers and non-traders. We study the LSM design in a dynamic setting from the perspective of a central bank or a system operator such as Payments Canada, Interac, Vocalink Mastercard, etc., including the optimal clearing time and amount. We take a systemic point of view and assume that the central bank has full information on the payments that need to be cleared and absolute control of the clearing decisions. Although central banks typically don't have such information and power in practice, it has been proposed that regulators should give central banks more power. The reasoning behind such a recommendation is to solve the payment system gridlock problem. This critical problem, which results from a coordination failure where banks delay their payment orders until they themselves receive incoming orders, has been studied in the literature (see, e.g., Afonso & Shin, 2011; Furfine & Stehm, 1998; Kahn et al., 2003; Rochet & Tirole, 1996). Our approach of a centralized decision maker is in line with other approaches in the literature, including free intraday credits provided by central banks (Kahn & Roberds, 2001; Martin, 2004) and regulatory frameworks that force the change of banks' settlement behavior (Maddaloni, 2015; Mills Jr & Nesmith, 2008; Nellen, 2019). Moreover, from a system optimization point of view, this fully centralized system provides a benchmark for the best that can be achieved.

We keep track of the pairwise debts between banks in the financial system as they arise. Such a network approach has been commonly taken in the literature to study the debt structure. The influential work by Eisenberg and Noe (2001) develops a general clearing model of financial systems to compute the clearing vector satisfying standard bankruptcy rules. It has inspired a large body of literature to study the contagion of default in financial systems, mostly using a static equilibrium model, including Amini et al. (2016a, 2016b), Capponi et al. (2016), Chen et al. (2016), Csóka and Herings (2021), and Khabazian and Peng (2019). Although our network model is similar at a high level, we do not consider the default of banks. We focus on the intraday clearing of debts (so default is not a major concern) and characterize the optimal dynamic clearing policy of the central bank that minimizes the clearing and liquidity costs.

This paper falls into the intersection of finance and operations. In particular, we adopt dynamic programming and borrow ideas from the theory of inventory management, two core research topics in Operations Research, and apply them to the debt clearing in a financial system. The goal is to introduce the dynamic management of resources to balance the debt over time. The optimal policy is also connected to the well-known (s, S)-policy in inventory management. Chapter 4.5 in Babich and Birge (2020) provides an excellent review of this approach. Birge et al. (2018) study the optimal portfolio allocation when the assets may default and the default events may cascade. They characterize the optimal feedback strategy in the optimal control framework. The debt generating process in our model is related to the interbank lending model in Capponi et al. (2020). They consider a number of banks in the system and their dynamic interactions through borrowing and lending. They characterize the stochastic process in the asymptotic sense when the number of banks tends to infinity, without any centralized control, which helps them analyze systemic risk indicators. The dynamic programming approach has also been used in optimal order executions (Chen et al., 2018; Tsoukalas et al., 2019). Similar to Giesecke and Kim (2011) and Sirignano and Giesecke (2019), we consider the dynamics of a large pool of collateralized debts. But our focus is on controlling the total debt level instead of analyzing the portfolio risk.

We consider the costs from the perspective of the central bank, including the direct costs of settling payments and the indirect costs of addressing risks and externalities associated with the settlement of payments (see Berger et al. (1996) for a comprehensive analysis of risks and costs in payment systems). The direct costs reflect the set-up costs of each clearing and the consumption of resources such as electronic communication and software. The indirect costs reflect the coordination effort of the central bank in each clearing and the liquidity costs resulting from the backlog of outstanding payment obligations. For the coordination effort, we note that the banks have incentives to delay the settlement of payments to maximize the return (Angelini, 1998; Bartolini et al., 2010; Bech & Garratt, 2003). From a systems point of view, settlement delay would reduce liquidity available for the payment system and pose a risk to the smooth operation of the payment system, i.e., cause gridlock, and the implementation of the optimal clearing policy. The coordination effort includes permitting daylight overdraft (Abbassi et al., 2017; Mills Jr, 2006), providing free intraday credit (Kraenzlin & Nellen, 2010; Martin, 2004) and/or charging penalty rates (Maddaloni, 2015; Nellen, 2019) to guarantee the timely settlement of payments. Therefore, the coordination costs in our model include the financial costs resulting from any default on the daylight overdraft or intraday credit and the investment costs of an enforcement technology that can effectively prevent a bank from defaulting (Furfine & Stehm, 1998; Martin, 2004; Mills Jr, 2006), and the costs of regulating penalty rates, such as the information costs for setting a proper penalty rate or designing a cost allocation scheme and the liquidity management costs incurred by banks to avoid penalty (Garratt, 2021; Maddaloni, 2015). For the liquidity costs, we note that when more payments are backlogged, there is higher uncertainty about the debt settlement by the end of the whole process. To ensure the settlement of the payments, the central bank requires banks to maintain a certain minimum amount of reserves to settle their debts (Heller & Lengwiler, 2003; Tomura, 2018). For those banks that are cautious about unsettled payments, they would hold excess reserves for self-insuring against the increasing payment shocks (Ashcraft et al., 2011). As reserves virtually provide no return, there is a financial cost for banks who could hold a high-return portfolio otherwise. The central bank is concerned about these financial costs because they represent a liquidity loss from settlement delays and an operational inefficiency of the payment system (Afonso & Shin, 2011; Kahn & Roberds, 2001). Moreover, banks incurring high financial costs may shift their payments to other cheaper and riskier systems, which is also a concern of the central bank (Berger et al., 1996; Capponi & Cheng, 2018; Holthausen & Rochet, 2006). We term the total financial costs resulting from the backlog (i.e. liquidity loss and market volume loss to the system) as the liquidity cost of uncleared payments.

2 The Clearing Model

2.1 Debt, Net Debt, Clearing and Costs

We consider the daily clearing problem of a central bank. The day is divided into $T < \infty$ periods. We start without relating quantities to their periods (we will add a time related subscript, $t = 1, \cdots, T$, later). There are N banks in the financial system. At the beginning of each period, the accumulated debt in dollar amount that bank i owes bank j is denoted by b_{ij} for $i, j = 1, \dots, N$. We use a matrix $\boldsymbol{B} = (b_{ij}) \in \mathbb{R}_+^{N \times N}$ to record the debt in the system. Note that $b_{ii} = 0$, which we impose throughout the paper, and b_{ij} and b_{ji} may be positive at the same time, that is, mutual debts may exist.

Based on the debt structure $\boldsymbol{B}$, we define the *net debt* of bank i as $b_i \triangleq \sum_{j=1}^{N}(b_{ij} - b_{ji})$. Note that the net debt represents the net debit position that could be either positive (debt) or negative (credit). We denote the accumulated debt in the system by $D \triangleq \sum_{i,j} b_{ij}$ and the total net debt in the system by $d \triangleq \sum_{i=1}^{N}(b_i)^+$, where x^+ represents the positive part. Because $\sum_{i=1}^{N} b_i = 0$, by definition, we also have $d = \sum_{i=1}^{N}(b_i)^-$, where $x^- = (-x)^+$. The net debt thus includes the outstanding debt in the system after all offsetting debts are canceled. We use *debt* to specifically indicate the accumulated debt, which is different from the *net debt* that is the debt net of offsetting debts.

Clearing. The central bank may decide to clear the debt in the system at the beginning of each period. That is, the debt bank i owes bank j changes from b_{ij} to $\tilde{b}_{ij}$. We use $\tilde{\cdot}$ to denote the debt structure right after clearing, for example, $\tilde{b}_{ij}$, $\tilde{b}_i$, $\tilde{d}$, $\tilde{D}$ and $\tilde{\boldsymbol{B}}$. We require the clearing to diminish the debt level of all banks. That is, either $b_i \geq \tilde{b}_i \geq 0$ or $b_i \leq \tilde{b}_i \leq 0$. In a more compact way, we write $|\boldsymbol{b}| \geq |\tilde{\boldsymbol{b}}|$ and $\tilde{\boldsymbol{b}}\boldsymbol{b} \geq 0$. Note that in this context clearing includes a range of activities of the central decision maker such as arranging, updating, coordinating and overseeing the settlement (i.e., the funds transfer) of payments among banks to control the debt level in the payment system.

Cost. As introduced in Section 1.1, the cost consists of three components. First, if clearing happens, then the debt in the system is reduced, i.e., $\tilde{D} < D$, and a *set-up cost* K is incurred. That is, $K\mathbb{1}_{D \neq \tilde{D}}$ where $\mathbb{1}.$ is the indicator function. In our model, this set-up cost reflects the fixed cost incurred for each clearing, such as the costs associated with electronic communication, staffing, and accounting as well as the discounted value of the one-time infrastructure investment required for clearing. Second, the *cost of coordinating* the clearing is proportional to the reduction of the net outstanding debt $d - \tilde{d}$, i.e., $\beta_c(d - \tilde{d})$. That is, the more net debt to clear, the higher coordination cost incurred by the system. Note that the cost doesn't include the cancellation of offsetting debt, which is negligible relatively because its cancellation only requires updating the balance sheets without any actual payment flows. We refer to the sum of the set-up and coordination costs as the *clearing cost*. Third, the *liquidity cost* in each period is proportional to the debt after clearing, i.e., $\beta_l \tilde{D}$. The liquidity cost captures the system inefficiency and the liquidity loss from the backlog and is proportional to the debt in the system after clearing. We emphasize that the liquidity cost is charged every period, even if clearing did not take place.

New debt. New debts are built up at the end of the period. The new debt bank i owes bank j, X_{ij}, is random and has distribution $F_{ij}(\cdot)$, which is independent of everything else. Note that the new debts, X_{ij}, incurred at each period are independent, but may not be identically distributed. As a result, at the beginning of the next period, the accumulated debt in dollar amount that bank i owes bank j is

$$\tilde{b}_{ij} + X_{ij}. \tag{1}$$

We let $\boldsymbol{X}$ denote the vector of realized debt.

We summarize the notation of our model in Table 1.

Notation	Definitions
T	number of periods in the planning horizon
N	number of banks in the payment system
$X_{ij,t} \in \mathbb{R}_+$	payment obligation of bank i to bank j incurred at the end of period t
$\mathbf{X}_t$	collection of $X_{ij,t}$ for all banks at period t, $\mathbf{X}_t \triangleq (X_{11,t}, \cdots, X_{NN,t}) \in \mathbb{R}_+^{N\times N}$
$X_t \in \mathbb{R}_+$	total payment obligation incurred at period t
$X_{i,t} \in \mathbb{R}$	net payment obligation of bank i incurred at period t
$b_{ij,t} \in \mathbb{R}_+$	payment obligation of bank i to bank j at the beginning of period t
$\mathbf{B}_t$	collection of $b_{ij,t}$ for all banks at period t, $\mathbf{B}_t \triangleq (b_{11,t}, \cdots, b_{NN,t}) \in \mathbb{R}_+^{N\times N}$
$\tilde{b}_{ij,t} \in \mathbb{R}_+$	payment obligation of bank i to bank j immediately after the clearing decision at period t
$D_t \in \mathbb{R}_+$	total debt level in the payment system at the beginning of period t
$\tilde{D}_t \in \mathbb{R}_+$	total debt level in the payment system immediately after the clearing decision at period t
$b_{i,t} \in \mathbb{R}$	net payment obligation of bank i at the beginning of period t
$\mathbf{b}_t$	collection of $b_{i,t}$ for all banks at period t, $\mathbf{b}_t \triangleq (b_{1,t}, \cdots, b_{N,t}) \in \mathbb{R}^N$
$\tilde{b}_{i,t} \in \mathbb{R}$	net payment obligation of bank i immediately after the clearing decision at period t
$\tilde{\mathbf{b}}_t$	collection of $\tilde{b}_{i,t}$ for all banks at period t, $\tilde{\mathbf{b}}_t \triangleq (\tilde{b}_{1,t}, \cdots, \tilde{b}_{N,t}) \in \mathbb{R}^N$
$d_t \in \mathbb{R}_+$	net debt level in the payment system at the beginning of period t
$\tilde{d}_t \in \mathbb{R}_+$	net debt level in the payment system immediately after the clearing decision at period t
β_l	unit liquidity cost of uncleared debts
β_c	unit coordination cost of clearing net debts
K	set-up cost of each clearing

Table 1: A summary of the notation used in the paper.

2.2 The Optimal Clearing Problem

After introducing the events in a single period, we formulate the dynamic problem to solve the optimal clearing policy. We count time backward and use the subscript t to denote the $(T-t)^{th}$ period, i.e., when there are t periods to go. In period t, the central bank observes the debt in the system $\boldsymbol{B}_t$, which is referred to as the state of the system. After clearing, the post-decision state is $\tilde{\boldsymbol{B}}_t$. The dynamics of the system is thus given by

$$b_{ij,t-1} = \tilde{b}_{ij,t} + X_{ij,t}.$$

The cost in period t is given by

$$K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l \tilde{D}_t.$$

We let $\Pi_t(\boldsymbol{B})$ be the minimum total cost at the beginning of period t when the state is $\boldsymbol{B}$. By the Bellman equation (Bertsekas et al., 2000), we have

$$\begin{aligned}
\Pi_t(\boldsymbol{B}_t) = \min_{\tilde{\boldsymbol{B}}_t} & \left\{ K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l \tilde{D}_t + \mathbb{E}\left[\Pi_{t-1}(\tilde{\boldsymbol{B}}_t + \boldsymbol{X}_t)\right]\right\} \\
s.t. \quad & \tilde{b}_{ii,t} = 0,\ \tilde{b}_{ij,t} \geq 0 && \text{(2a)} \\
& b_{i,t} = \sum_{j=1}^{N}(b_{ij,t} - b_{ji,t}),\ \tilde{b}_{i,t} = \sum_{j=1}^{N}(\tilde{b}_{ij,t} - \tilde{b}_{ji,t}) && \text{(2b)} \\
& d_t = \sum_{i=1}^{N}(b_{i,t})^+,\ \tilde{d}_t = \sum_{i=1}^{N}(\tilde{b}_{i,t})^+ && \text{(2c)} \\
& D_t = \sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij,t},\ \tilde{D}_t = \sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{b}_{ij,t} && \text{(2d)} \\
& |\boldsymbol{b}_t| \geq |\tilde{\boldsymbol{b}}_t|,\ \tilde{\boldsymbol{b}}_t \cdot \boldsymbol{b}_t \geq 0 && \text{(2e)}
\end{aligned}$$

where the expectation is taken with respect to $\boldsymbol{X}_t$ whose elements have distribution $F_{ij}(\cdot)$. The objective of (2) is the sum of the cost incurred in the current period and the optimal cost-to-go starting from the next period. The constraints of (2e) capture the fact that the debt after clearing cannot exceed that before clearing, which are introduced in Section 2. All debts are cleared at the end of the whole process, i.e., in period $t = 0$. By convention, we let $\Pi_0(\boldsymbol{B}) \equiv 0$ for any $\boldsymbol{B}$.

Remark 1. *The clearing model* (2) *mainly simplifies two aspects of payment systems in practice such as Large Value Transfer System (LVTS) of Payments Canada. First, we assume a volume-based clearing cost in* (2)*, while LVTS implements a transaction-based flat cost. Second, we assume the central bank or the payments company has absolute control in clearing debts, while banks in practice may delay the settlement of their payments to increase the opportunity of offsetting and reduce the cost of posting collateral. These simplifications make the optimization more tractable.*

3 The Optimal Clearing Policy

3.1 State Space Reduction

Note that (2) is quite intractable both theoretically and computationally, because both the state space ($\boldsymbol{B}$) and the action space ($\tilde{\boldsymbol{B}}$) are N^2-dimensional. In this section, we apply a state space reduction that significantly simplifies the analysis. In particular, consider a function $V_t(\cdot)$ defined on $(b_1, \ldots, b_{N-1}, D) \in \mathbb{R}^{N-1} \times \mathbb{R}_+$. We show that

Proposition 3.1 (Value function of the reduced DP)**.** *There exist a unique sequence of functions* $V_t\colon \mathbb{R}^{N-1} \times \mathbb{R}_+ \to \mathbb{R}$, $t = 0, 1, \ldots, T$, *such that* $\Pi_t(\boldsymbol{B}) = V_t(b_1, \ldots, b_{N-1}, D)$, *where* $\boldsymbol{B} = (b_{ij})_{i,j=1}^N$, $b_i = \sum_{j=1}^N (b_{ij} - b_{ji})$ *and* $D = \sum_{i,j} b_{ij}$.

Proposition 3.1 states that the value functions Π_t can be represented by another function of a much lower dimension. In particular, it only depends on the net debt of all banks in the system $(b_1, \ldots, b_N)$ and the total debt in the system D, but *not* the pairwise debt relationship between the banks. For the ease of notation, we use the vectorized version $V_t(\boldsymbol{b}, D)$, under the convention that $\boldsymbol{b}$ satisfies $\sum_{i=1}^N b_i = 0$. The reduction implies that the problem may have a rich implicit structure that can be exploited to simplify the analysis.

However, this reduction alone does not allow us to reformulate the Bellman equation in a more compact form, because the optimal policy $\tilde{\boldsymbol{B}}^*$ in (2) may still depend on $\boldsymbol{B}$. The following result states that the function V itself is the value function of a proper dynamic program. It serves as the building block for the subsequent analysis.

Proposition 3.2 (Bellman equation of the reduced DP)**.** *The functions* V_t *uniquely defined in Proposition 3.1 satisfy*

$$V_t(\boldsymbol{b}_t, D_t) = \min_{\tilde{D}_t, \tilde{\boldsymbol{b}}_t} \left\{ K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l \tilde{D}_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})\right] \right\}$$

$$\text{s.t.} \quad \sum_{i=1}^N b_{i,t} = \sum_{i=1}^N \tilde{b}_{i,t} = 0 \tag{3a}$$

$$d_t = \sum_{i=1}^N (b_{i,t})^+, \ \tilde{d}_t = \sum_{i=1}^N (\tilde{b}_{i,t})^+ \tag{3b}$$

$$|\boldsymbol{b}_t| \geq |\tilde{\boldsymbol{b}}_t|, \ \tilde{\boldsymbol{b}}_t \cdot \boldsymbol{b}_t \geq 0, \ \tilde{D}_t \geq \tilde{d}_t \tag{3c}$$

$$b_{i,t-1} = \tilde{b}_{i,t} + \sum_{j=1}^N (X_{ij,t} - X_{ji,t}), \ D_{t-1} = \tilde{D}_t + \sum_{i=1}^N \sum_{j=1}^N X_{ij,t}, \tag{3d}$$

for $t = 1, \ldots, T$. *Moreover, for any* $\boldsymbol{B}_t$ *such that* $b_{i,t} = \sum_{j=1}^N (b_{ij,t} - b_{ji,t})$, $D_t = \sum_{i=1}^N \sum_{j=1}^N b_{ij,t}$ *and* $\Pi_t(\boldsymbol{B}_t) = V_t(\boldsymbol{b}_t, D_t)$, *there exists an optimal solution* $\tilde{\boldsymbol{B}}_t^* = (\tilde{b}_{ij,t}^*)_{i,j=1}^N$ *to* (2) *that satisfies* $\tilde{b}_{i,t}^* = \sum_{j=1}^N (\tilde{b}_{ij,t}^* - \tilde{b}_{ji,t}^*)$ *and* $\tilde{D}_t^* = \sum_{i=1}^N \sum_{j=1}^N \tilde{b}_{ij,t}^*$, *where* $(\tilde{\boldsymbol{b}}_t^*, \tilde{D}_t^*)$ *is an optimal solution to* (3).

We discuss the implications of Proposition 3.2 below. As implied by Proposition 3.1, the function V can be regarded as the optimal cost in which the central bank only needs to keep track of the net debt of individual banks and the total debt in the system. The Bellman equation (3) introduces a reformulation of (2) because of the reduced state. For example, (3a) imposes that the total net debt sums up to zero, which always needs to hold. Equations (3b) and (3c) echo the same constraints in (2c) and (2e), while $\tilde{D}_t \geq \tilde{d}_t$ follows directly from the definition of $\tilde{D}_t$ by (2d) and the definition of $\tilde{d}_t$ by (2b) and (2c). Therefore, we can focus on V_t and the corresponding simplified dynamic program independently of the original formulation (2). Moreover, Proposition 3.2 also allows "reverse-engineering" of its solution, i.e., getting a solution for the original DP. Specifically, Proposition 3.2 inspires Algorithm 1 (see below) that for any optimal solution to the reduced value function V_t reconstructs an optimal solution to (2), as illustrated in Example 1. Therefore, in practice, one can simply focus on (3). In particular, the central bank doesn't need to track the mutual debt between all pairs of members; instead, it only requires each bank to report their net debt position and keep track of the total debt in the system. This lower data requirement provides significant practical values since many financial systems do not require banks to report the pairwise debt but only the aggregate (net) debt.

Algorithm 1 Implementing the clearing policy solved from the reduced dynamic program (3)

Require: N, the net and outstanding debts before the clearing $(\boldsymbol{b}, D)$, the optimal policy $(\tilde{\boldsymbol{b}}, \tilde{D})$ instructed by solving (3)

Initialize: $\tilde{\boldsymbol{B}} \leftarrow 0^{N\times N}$, remaining banks $S \leftarrow \{1,\dots,N\}$, $\tilde{d} = \sum_{i=1}^{N}(\tilde{b}_i)^+$
while $|S| > 2$ and $\tilde{d} > 0$ **do**
  $i_1 \leftarrow \arg\max\{\tilde{b}_i | \tilde{b}_i \leq 0, i \in S\}$, $i_2 \leftarrow \arg\min\{\tilde{b}_i | \tilde{b}_i > 0, i \in S\}$
  if $|\tilde{b}_{i_1}| \leq |\tilde{b}_{i_2}|$ **then**
    $\tilde{b}_{i_2 i_1} \leftarrow |\tilde{b}_{i_1}|$, $\tilde{b}_{j i_1} \leftarrow 0$, $\tilde{b}_{i_1 j} \leftarrow 0$ for all $j \neq i_2$ ▷ Bank i_1 only engages with i_2, which owes $|\tilde{b}_{i_1}|$
    $\tilde{d} \leftarrow \tilde{d} - |\tilde{b}_{i_1}|$, $\tilde{b}_{i_2} \leftarrow \tilde{b}_{i_2} - |\tilde{b}_{i_1}|$, $\tilde{D} \leftarrow \tilde{D} - |\tilde{b}_{i_1}|$, $S \leftarrow S \setminus \{i_1\}$ ▷ Remove i_1 from the system
  else
    $\tilde{b}_{i_2 i_1} \leftarrow |\tilde{b}_{i_2}|$, $\tilde{b}_{j i_2} \leftarrow 0$, $\tilde{b}_{i_2 j} \leftarrow 0$ for all $j \neq i_1$ ▷ Bank i_2 only engages with i_1, to which it owes $|\tilde{b}_{i_2}|$
    $\tilde{d} \leftarrow \tilde{d} - |\tilde{b}_{i_2}|$, $\tilde{b}_{i_1} \leftarrow \tilde{b}_{i_1} + |\tilde{b}_{i_2}|$, $\tilde{D} \leftarrow \tilde{D} - |\tilde{b}_{i_2}|$, $S \leftarrow S \setminus \{i_2\}$ ▷ Remove i_2 from the system
  end if
end while
$i_1 \leftarrow \arg\min\{\tilde{b}_i | i \in S\}$, $i_2 \in S \setminus \{i_1\}$ ▷ Two banks remaining in S or zero net debts $\tilde{d} = 0$
$\tilde{b}_{i_1 i_2} \leftarrow (\tilde{D} - |\tilde{b}_{i_1}|)/2$, $\tilde{b}_{i_2 i_1} \leftarrow (\tilde{D} + |\tilde{b}_{i_1}|)/2$ ▷ Allocate the remaining debt to the two banks
return $\tilde{\boldsymbol{B}} = (\tilde{b}_{ij})$

Example 1 (**Potential clearing policies and associated costs**). *Consider three banks ($N = 3$). At the beginning of the period, their debts are shown in Table 2. We consider four options to clear the debt of the banks, illustrated by Table 3. Options one and two clear all offsetting debts (because $\tilde{D} = \tilde{d}$) and \$50 net debt. While option one clears \$50 net debt owed by Bank 1 to Bank 3,* *option two clears \$50 net debt owed by Bank 2 to Bank 3. These options demonstrate that stating the amount to clear is insufficient to characterize a unique system state. Options three and four correspond to no-clearing and clearing-all-debts, respectively.*

Table 4 shows the outcome after clearing the debts using option one. Note that when all the offsetting debts are cleared, there are many possible ways to clear the net debts, such as options one and two. When the net debts after clearing are determined, the clearing outcome is unique. Under option one, if we set $K = 0$, $\beta_l = 1$ and $\beta_c = 0.5$, then the total cost in that period is $K + \beta_c(d - \tilde{d}) + \beta_l \tilde{D} = 125$.

b_{ij}	Bank 1	Bank 2	Bank 3	$\sum_j b_{ij}$
Bank 1	0	200	300	500
Bank 2	150	0	250	400
Bank 3	300	100	0	400
$\sum_i b_{ij}$	450	300	550	$D = 1300$
b_i	50	100	-150	$d = 150$

Table 2: Debts before clearing

	$\tilde{b}_1$	$\tilde{b}_2$	$\tilde{b}_3$	$\tilde{d}$	$\tilde{D}$
Option 1	0	100	-100	100	100
Option 2	50	50	-100	100	100
Option 3	0	0	0	0	0
Option 4	50	100	-150	150	1300

Table 3: Potential clearing options

$\tilde{b}_{ij}$	Bank 1	Bank 2	Bank 3	$\sum_j \tilde{b}_{ij}$
Bank 1	0	0	0	0
Bank 2	0	0	100	100
Bank 3	0	0	0	0
$\sum_i \tilde{b}_{ij}$	0	0	100	$\tilde{D} = 100$
$\tilde{b}_i$	0	100	-100	$\tilde{d} = 100$

Table 4: Debts after clearing using option one

3.2 Structural Properties

In this section, we provide a few structural properties of the optimal clearing policy. We note that in any period t, there are a few policy regimes that may be interesting to the central bank if it decides to clear debts and incurs the set-up cost in that period:

- The clear-all-offsetting policy: clear all the offsetting debts first and then proceed to net debts. Since the net debt before clearing is represented by the vector $\boldsymbol{b}_t$, only clearing the offsetting debt does not affect $\boldsymbol{b}_t$ or d_t. When all the offsetting debts are cleared, we have $\tilde{D}_t = \tilde{d}_t$. Therefore, after all offsetting debts are cleared, we have $\tilde{d}_t = \tilde{D}_t = d_t$. If in addition the net debt is also cleared (partially), then we have $\tilde{d}_t = \tilde{D}_t < d_t$.

- The clear-all policy: clear all the debt. In this case, we have $\tilde{d}_t = \tilde{D}_t = 0$.

Note that the clear-all policy is a special case of the clear-all-offsetting policy, which is a special family of clearing policies. In general, the central bank may choose to partially clear the offsetting debts or the net debts. However, we next establish that whenever a clearing occurs, the central bank should clear all offsetting debts, and when the liquidity cost outweighs the coordination cost, it should clear all debts. We first present a technical lemma.

Lemma 3.3 (Continuity and monotonicity of the value function)**.** *(a) For* $t = 1, \ldots, T$ *and any* $u \geq 0$*, we have* $V_t(\boldsymbol{b}'_t, D_t) - V_t(\boldsymbol{b}_t, D_t) \leq \beta_c u$ *if* $\sum_{i=1}^N (b'_{i,t} - b_{i,t})^+ \leq u$*. (b) For* $t = 1, \ldots, T$ *and any* D'_t, D_t *with* $D'_t \geq D_t$*, we have* $0 \leq V_t(\boldsymbol{b}_t, D'_t) - V_t(\boldsymbol{b}_t, D_t) \leq t\beta_l(D'_t - D_t)$*.*

Lemma 3.3 part (a) shows the value function $V(\boldsymbol{b}, D)$ is continuous in its first argument. Lemma 3.3 part (b) shows the value function $V(\boldsymbol{b}, D)$ is continuous and increasing in its second argument. Given the total debt, the value function is mainly determined by the coordination cost of the net debts. As the net debts can be either positive or negative and may be offset in later periods, a larger net debt does not necessarily result in a higher coordination cost. Lemma 3.3 part (a) bounds the difference in the value function caused by given differences in the net debts. Moreover, the newly created net debts are directly counteracted in the boundary condition, i.e. $\sum_{i=1}^N (b'_{i,t-1} - b_{i,t-1})^+ = \sum_{i=1}^N (\tilde{b}'_{i,t} + x_{i,t} - (\tilde{b}_{i,t} + x_{i,t}))^+ = \sum_{i=1}^N (\tilde{b}'_{i,t} - \tilde{b}_{i,t})^+ \leq u$. Thus, Lemma 3.3 part (a) can be easily extended to the next period, which, using induction, will help prove that offsetting debts should be cleared before non-offsetting ones. Lemma 3.3 part (b) bounds the difference in the value function caused by a given difference in the total debt and will help prove the optimality of the clear-all-offsetting policy. This lemma also helps analyze the optimal clearing policy when the total debt is identical to the net debt in the system.

Next, we argue that if the clearing process is conducted in a period, i.e., the set-up cost K is incurred, then it is optimal to clear *all the offsetting debt* first. Note that clearing the offsetting debt does not affect the net debt $\boldsymbol{b}$ and only decreases D. Mathematically, if there is no clearing or only some of the offsetting debts are cleared, then we have $\tilde{d}_t = d_t$; if all the offsetting debts are cleared as well as some net debts, then we have $\tilde{d}_t = \tilde{D}_t$. The next result shows that whenever clearing, it is optimal to clear all offsetting debt.

Proposition 3.4 (Clear-all-offsetting is optimal)**.** *The optimal solution* $(\tilde{\boldsymbol{b}}_t^*, \tilde{D}_t^*)$ *to* $V_t(\boldsymbol{b}_t, D_t)$ *in* (3) *must satisfy either* $\tilde{D}_t^* = D_t$ *or* $\tilde{d}_t^* = \tilde{D}_t^* \leq d_t$*, i.e., not clearing or clearing all offsetting debt.*

Note that this result is far from obvious. Since we do not specify the distribution of the new debts created in each period, $\boldsymbol{X}_t$, we do not rule out the possibility that in expectation, large mutual debts between banks i and j are going to be created in the next period. One may expect to leave some offsetting debts in the system as buffers and clear other debts first. Our result points out that whenever some clearing occurs such a policy is not optimal.

When the liquidity cost outweighs the coordination cost, we show a much stronger result for the optimal clearing policy. In particular, when the central bank clears, it is optimal to clear *all* debts. This setting can occur when settlement delays, such as operational outages and wide-scale disruptions, are significant (Bech & Garratt, 2012) and the coordination costs are low, e.g., when the central decision maker generally has an advantage in supplementing intraday liquidity and enforcing the settlement of payments (Abbassi et al., 2017; Kahn & Roberds, 2009; Kraenzlin & Nellen, 2010). This setting implies that more emphasis should be put on the immediacy of settling payments than on the opportunity of offsetting. Such settings are sensible when the set-up cost is low (e.g., due to technological developments) and may account for the evolution of payment systems from DNS to RTGS.

Proposition 3.5 (Clear-all is optimal)**.** *If* $\beta_l \geq 2\beta_c$*, then it is optimal to not clear at all or clear all debts* ($\tilde{d}_t^* = \tilde{D}_t^* = 0$) *in each period. Moreover, the clearing region of* $(\boldsymbol{b}_t, D_t)$ *satisfies:*

1. *If* $D_t = d_t < \frac{K}{t\beta_l}$*, then it is optimal not to clear.*

2. *If* $d_t \geq \frac{K}{\beta_l - 2\beta_c}$*, then it is optimal to clear all debts.*

3. *If it is optimal to clear for state* $(\boldsymbol{b}_t, D_t)$*, then it is optimal to clear for* $(\boldsymbol{b}'_t, D_t + d'_t - d_t)$*, where* $|b_{i,t}| \leq |b'_{i,t}|$ *and* $b_{i,t}b'_{i,t} \geq 0$ *for all* i*.*

4. *For a given* $\boldsymbol{b}_t$*, it is optimal to clear for state* $(\boldsymbol{b}_t, D_t)$ *when* $D_t \geq g(\boldsymbol{b}_t)$*, where the threshold* $g_t(\cdot)$ *satisfies* $\sum_{i=1}^N (x_i)^+ \leq g_t(\boldsymbol{x}) \leq \frac{K}{\beta_l} + \frac{2\beta_c}{\beta_l} \sum_{i=1}^N (x_i)^+$.

Proposition 3.5 provides a simpler structure of the optimal clearing policy. In fact, the clearing policy boils down to a binary decision: is it optimal to clear (all the debt) or not in the current period. Therefore, the central bank doesn't need to worry about the amount of debts to clear and just focuses on the timing. Proposition 3.5 also characterizes the clearing region of the state space. In general, it is optimal to clear when the total net debt and the total debt are large.

We remark that the condition $\beta_l \geq 2\beta_c$ is sufficient but not necessary for the optimality of the clear-all policy. Consider the following examples and Section 3.3 for situations in which it is optimal to clear all debts even when this condition doesn't hold. Still, the condition points out qualitatively that when the liquidity cost is large relative to the coordination cost, then the central bank should use this type of policy.

Example 2 (**Optimality of clear-all without sufficient conditions**). *Consider three banks ($N = 3$) and two periods ($T = 2$). Let $K = 30$, $\beta_l = 1$ and $\beta_c = 1$. At the beginning of period two, the net debts are respectively $b_{1,2} = 80, b_{2,2} = 100, b_{3,2} = -180$, so the total net debt is $d_2 = 180$. Further we let the total debt be $D_2 = 380$. The new net payment obligations created in period two are: $x_{1,2} = 30, x_{2,2} = 20, x_{3,2} = -50$, with a total new debt of 200. As $K = 30 < \beta_l(D_2 - d_2) = 200$, it is optimal to clear in period two.*

Case one – Clear all debts: the cost in period two is $K + \beta_c d_2 = 30 + 180 = 210$. So that the net debts at the beginning of period one are the net new payments created in period two: $b_{1,1} = 30, b_{2,1} = 20, b_{3,1} = -50$ and $D_1 = 200$.

Case two – Clear but leave some net debt: here we clear all offsetting debt, but leave a balance of d^c to the next period. Then, the cost in period two is $K + \beta_c(d_2 - d^c) + \beta_l d^c = 210$. Suppose $\tilde{b}_{1,2} = d^c, \tilde{b}_{2,2} = 0, \tilde{b}_{3,2} = -d^c$ and $\tilde{D}_2 = d^c$ where $0 \leq d^c \leq 80$. The debts at the beginning of period one are: $b_{1,1} = 30 + d^c, b_{2,1} = 20, b_{3,1} = -50 - d^c$ and $D_1 = 200 + d^c$.

As the set-up cost is lower than the liquidity cost of offsetting debts and $V_0 = 0$, it is optimal to clear all debts in period one in both cases. In Case one, the cost in period one is $K + \beta_c d_1 = 30 + 50 = 80$. In Case two, the cost in period two is $K + \beta_c d_1 = 30 + 50 + d^c = 80 + d^c$. Therefore, it is optimal to clear all, i.e., setting $d^c = 0$.

Example 3 (**Optimality of clear-all only with sufficient conditions**). *Following Example 2, we show that clear-all may not be optimal when the sufficient conditions are not satisfied. All else being equal, assume the new net payment obligations created in period two are: $x_{1,2} = -50, x_{2,2} = 0, x_{3,2} = 50$, with a total new debt of 200.*

As $K < \beta_l(D_2 - d_2)$, it is optimal to clear in period two. In Case one, the cost in period two is $K + \beta_c d_2 = 30 + 180 = 210$. The debts at the beginning of period one are: $b_{1,1} = -50, b_{2,1} = 0, b_{3,1} = 50$ and $D_1 = 200$. In Case two, the cost in period two is $K + \beta_c(d_2 - d^c) + \beta_l d^c = 210$. The debts at the beginning of period one are: $b_{1,1} = -50 + d^c, b_{2,1} = 0, b_{3,1} = 50 - d^c$ and $D_1 = 200 + d^c$.

As the set-up cost is lower than the liquidity cost of offsetting debts and $V_0 = 0$, it is optimal to clear all debts in period one in both cases. In Case one, the cost in period one is $K + \beta_c d_1 =$

$30 + 50 = 80$. *In Case two, the cost in period one is* $K + \beta_c d_1 = 30 + 50 - d^c = 80 - d^c$. *Therefore, it is optimal to clear but leave some net debt, i.e., setting* $d^c > 0$.

Next, we show that clear-all is optimal for the above debt process when the sufficient conditions are satisfied by letting $\beta_c = 0.4 < 2\beta_l$. *As* $K < \beta_l(D_2 - d_2)$, *it is optimal to clear in period two. In Case one, the cost in period two is* $K + \beta_c d_2 = 102$. *In Case two, the cost in period two is* $K + \beta_c(d_2 - d^c) + \beta_l d^c = 102 + 0.6d^c$.

As the set-up cost is lower than the liquidity cost of offsetting debts and $V_0 = 0$, *it is optimal to clear all debts in period one in both cases. In Case one, the cost in period one is* $K + \beta_c d_1 = 30 + 25 = 55$. *In Case two, the cost in period one is* $K + \beta_c d_1 = 30 + 25 - 0.5d^c = 55 - 0.4d^c$. *Therefore, it is optimal to clear all, i.e., setting* $d^c = 0$.

3.3 Special Cases

In this section, we study two special cases: two banks and deterministic debt processes. Studying these cases allows us to derive richer structures of the optimal policy that shed light on the general case.

Two Banks

When $N = 2$, the system's state can be represented by two variables, $(b_{1,t}, D_t)$, in period t. By the results in Section 3.2, the optimal clearing policy always clears all offsetting debt first, if clearing is necessary at all. Moreover, if $\beta_l \geq 2\beta_c$, it is optimal to clear all debts in the system when clearing is needed. Both structures describe how to optimally clear the debts, while it is still unclear *when* it is necessary to clear. With two banks, we characterize the boundary that determines whether the central bank should clear the debt in a period.

Corollary 3.6 (Two banks)**.** *(a) For any* t, *there is a function* $g_t(\cdot)$ *such that* $g_t(x) \geq x$ *and for any state* $(b_{1,t}, D_t)$, *it is optimal to clear if and only if* $D_t \geq g_t(|b_{1,t}|)$. *(b) In addition, when* $\beta_l > 2\beta_c$, $g_t(\cdot)$ *is an increasing function such that (i)* $g_t(x) > x$ *for* x *less than a threshold and (ii)* $g_t(x) = x$ *otherwise.*

By Corollary 3.6 the function $D = g_t(b)$ divides the space of (b, D) into two regions, as shown in Figure 1. When the state $(b_{1,t}, D_t)$ falls into the top-left region, it is optimal to clear. That is, the outstanding debt D_t is large relative to the net debt d_t, or equivalently, the net debt is small relative to D_t. Note that without $\beta_l > 2\beta_c$, i.e., part (a) of Corollary 3.6, the boundary may touch $D = |b|$ multiple times and $\tilde{D}^*$ may be positive. Note that once $g_t(x) = x$, it implies that for the net debt level $|b_{1,t}| = x$, it is always optimal to clear regardless of the total debt. Figure 1 also implies that when the net debt or the total debt is sufficiently large, then it is optimal for the central bank to clear the debt.

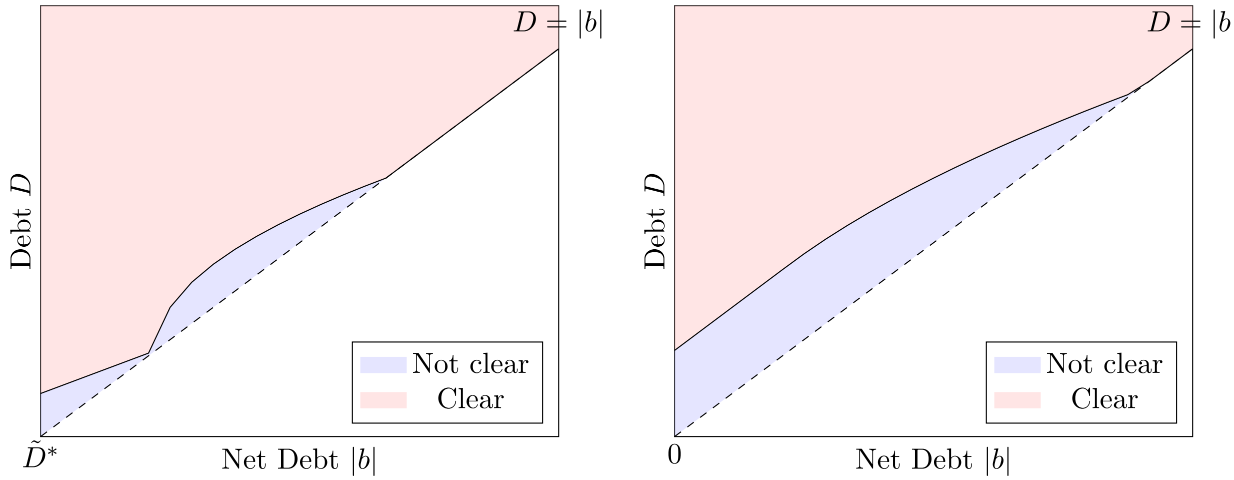

Figure 1: The region that is optimal to clear. The left panel shows part (a) of Corollary 3.6 and the right panel shows part (b).

It is worth noticing that for fixed total debt D, an increase in the net debt position, measured by $|b|$ (or d for more than two banks), may switch the optimal policy from clear to not clear. As illustrated by Figure 1, it may be more cost-efficient not to clear as $|b|$ increases for given D. To understand the intuition, note that the liquidity cost is incurred due to the total debt, not the net debt. In fact, clearing a large position of net debt incurs a high coordination cost. In some situations, if the total debt in the system is manageable while the net debt level is high, then the central bank may delay clearing and wait for the net debt to be offset by new debts generated in the near future.

Deterministic Debt Processes

In this section, we consider the scenario where the generated debts are deterministic and stationary over time. Let $x_i = \sum_{j=1}^{N}(x_{ij} - x_{ji})$ be the net debt of bank i in each period, $y = \sum_{i=1}^{N}(x_i)^+$ be the total net debt per period, and $z = \sum_{i,j} x_{ij}$ be the total debt per period. When $\beta_l \geq 2\beta_c$, by Proposition 3.5 the optimal clearing policy always clears all debts if it clears any debt in a period. In fact, as we show next, we can relax the condition to $\beta_l \geq \beta_c$ for deterministic debt processes. We seek to find the clearing pattern of the optimal policy when the system starts with no debts, i.e., $D_T = d_T = 0$.

Proposition 3.7 (Deterministic debt process)**.** *When $\beta_l \geq \beta_c$, it is optimal to clear all the debt or no debt in each period. The optimal clearing policy consists of at most two types of clearing* *cycles with lengths τ and $\tau + 1$. Moreover, having all clearing cycles of length $\tau^* = \left\lfloor \sqrt{\frac{2K}{\beta_l z}} \right\rfloor$ is asymptotically optimal as $T \to \infty$.*

We consider $T \to \infty$ for theoretical and practical reasons. Theoretically, it allows us to ignore the effect of the remainders when T is not a multiple of the cycle length and focus on the simplified version of the optimal cycle length independently of T. In practice, since a period in the model represents an interval of 5 to 30 minutes by which frequency the central bank monitors the debt in the system, it is reasonable to assume that the horizon (a day) has many such periods.

Proposition 3.7 provides two insights. First, the cost can roughly be interpreted as a convex function of the cycle length. Therefore, it is optimal to keep the clearing cycles of similar length, i.e., it is more cost efficient to have two medium clearing cycles than a long and a short cycle. This is similar to the EOQ model. However, due to the discrete time periods, the cycle length is not necessarily uniform but may have two consecutive values. This observation largely holds for the stochastic model in our numerical experiments as well. Second, the optimal cycle length as $T \to \infty$ is proportional to $\sqrt{K}$, the square root of the set-up cost. It is also proportional to $1/\sqrt{\beta_l z}$, i.e., a larger total new debt per period or a higher liquidity cost leads to more frequent clearings. Surprisingly, it doesn't depend on β_c or the net debt per period. Intuitively, note that the net debt in the system increases linearly without clearing. Therefore, the coordination cost when clearing occurs is $\beta_c y \tau$ for cycle length τ. It implies that the average coordination cost per period is independent of the cycle length and thus the optimal cycle length τ^* as well.

Although we study the deterministic stationary debt process in this section, the insights may be applied to the general stochastic and nonstationary case in several ways. First, if the random debt process has a small variance relative to the mean, that is, the payments between banks largely follow similar patterns over days and can be accurately forecast, then the optimal clearing policy derived in Proposition 3.7 may be near optimal. This policy is easy to compute and implement and requires only the first-order information (the mean) of the debt process compared to the optimal policy based upon the dynamic programming. Second, in applications, the debt process is often nonstationary. For example, well-documented in the literature, the debts are typically generated more frequently when the market opens and at the end of the day. We may divide a day into separate regimes so that in each regime the debt process is approximately stationary and apply the optimal clearing cycle accordingly. For example, a day can be divided into three regimes: market opening, mid of the day, and market closing. Thanks to the "renewal" of the system after each clearing, such a simple strategy may lead to near-optimal performance if the length of the optimal clearing cycle is short relative to the duration of the regimes. If the process is nonstationary and volatile, then regime-switching models may be used (Cai et al., 2019), which is out of the scope of this paper.

4 Numerical Experiments

Here we first use simulated data to compare the performance of four policies: clearing all debts in each period (Policy 1), clearing all debts only in the last period (Policy 2), the fluid policy in Proposition 3.7 (Policy 3), and the optimal clearing policy. All experiments are performed in Python 3.9.6 on a server with an Intel(R) Xeon(R) Gold 6240 CPU (2.60GHz) and 256 GB RAM running CentOS 7.9. We then present a case study based upon data from Payments Canada.

4.1 Comparison of Clearing Policies

In this section, we present numerical experiments to illustrate the effects of the clearing policy on various performance measures. We consider three cost measures and three non-cost measures. The cost measures are the total cost (the objective of the optimal policy), the total clearing cost (set-up plus coordination costs), and the total liquidity cost. The non-cost measures are the average clearing time, the maximum debt level, and the average backlog of Bank 1. More precisely, we define the average clearing time as the average time that payment obligations stay in the system from being generated to being cleared. We assume that debts are cleared on a first-in-first-out basis, and calculate the average clearing time as the ratio of the total outstanding debt in the system to the total generated debt in the process, i.e.,

$$\delta = \sum_{i,j,t} \tilde{B}_{ij,t} / \sum_{i,j,t} X_{ij,t}. \tag{4}$$

The maximum debt level of the system is

$$M = \max_t \left(\sum_{i,j} \tilde{B}_{ij,t} \right). \tag{5}$$

The average backlog of Bank 1 is

$$H = \sum_{j,t} \tilde{B}_{1j,t} / T. \tag{6}$$

The three non-cost measures provide important information for the central bank other than the cost. For example, the average clearing time is a well-understood indicator for system efficiency (Bartolini et al., 2010; Byck & Heijmans, 2021). The maximum debt level reflects the systemic risk (Mills Jr & Nesmith, 2008). The average backlog is correlated with the risk exposure of individual banks (Afonso & Shin, 2011). Note that when all the debts are cleared every period, the three non-cost measures, (4) to (6), are always zero.

In the following, we consider 5 banks ($N = 5$) and 16 periods ($T = 16$), which can be thought of as working hours for the payment system. We compare, under various cost parameters, the performance of the optimal clearing policy with the following three benchmark policies: Policy 1: clearing all debts in each period, Policy 2: clearing all debts only in the last period, and Policy 3: the optimal policy in Proposition 3.7 associated with the deterministic approximation of the debt processes. Policy 1 is similar to RTGS that clears debt in real time, which is gaining popularity in the industry. Policy 2 is currently adopted by many clearing and settlement systems such as the Large Value Transfer System (LVTS) of Payments Canada, which keeps track of the offsetting debts in each period and clears all the debt only at the end of the day. Policy 3 is the heuristic policy solved in Section 3.3 when treating the debt process as deterministic, which is very computationally efficient.

We consider three debt dynamics: For Debt Process I (the symmetric case), all the bilateral debts $X_{ij,t}$ have i.i.d. normal distribution with mean 0 and standard deviation 0.2, left truncated at 0. For Debt Process II (the asymmetric case), all the bilateral debts are independent. The debts owed by Bank 1 have truncated normal distribution with mean 0.2 and standard deviation 0.2. All the other debts have truncated normal distribution with mean 0 and standard deviation 0.2. By creating this process, we mimic the scenario in which one bank tends to carry excessive debt over time. For Debt Process III (the correlated case), the bilateral debts in each period follow a multivariate normal distribution with mean vector $\mathbf{0}$ and covariance matrix $\mathbf{\Sigma}$, left truncated at $\mathbf{0}$. Given the eigenvalue vector $\mathbf{1}$, the covariance matrix $\mathbf{\Sigma}$ is randomly generated following a numerically stable algorithm by Davies and Higham (2000) and scaled to have $\mathbf{0.04}$ on the diagonal. By Debt Process III, we consider a scenario that the debts have correlations and may lead to fluctuations of the net debts.

For the optimal policy, to solve the reduced dynamic program in Proposition 3.2 numerically, we discretize the state space to $\{-3, -1, \ldots, 3\}$ for the net debts ($\{-3, -1, \ldots, 5\}$ for the net debt of Bank 1 in Debt Process II) and $\{0, 1, \ldots, 20\}$ for the total debt. To capture that each day starts afresh, we set the initial debts to zero in the system. We compare the three clearing policies under six sets of cost parameters. We generate 3×10^7 random bilateral debt matrices based on the debt dynamics above. With Monte Carlo simulation, we calculate the transition probability matrix of the state, the net debts and the total debt, in the reduced dynamic program, after rounding the simulated state dynamics to the nearest integers. We then solve the dynamic program numerically using backward induction as the states have been discretized.

Table 5 depicts the performance of the clearing policies under the three cost measures. All measures are calculated using the average of 100 simulated sample paths and we also provide their

95% confidence interval. By design, Policy 1 minimizes the liquidity cost by clearing in every period, incurring a large clearing cost, and Policy 2, to the opposite, leaves the debt in the system until the end of the day and causes a large liquidity cost. Policy 3 trades off both costs in a heuristic fashion. By appropriately trading off the liquidity cost L with the clearing cost C, the cost of the optimal policy is significantly lower than these of Policy 1 and 2. On average, the cost of Policy 1 is 25.97% higher than the optimal cost (in the range of 6.17% and 109.79%), the cost of Policy 2 is 2325.66% higher than the optimal cost (in the range of 199.10% and 11200.73%), and the cost of Policy 3 is 6.23% higher than the optimal cost (in the range of 0 and 24.96%).

As expected, the benefit of the optimal policy over Policy 1 is most pronounced when the set-up cost increases or the unit liquidity cost decreases relative to the unit clearing cost. The debt processes may also affect the trade-off. For example, in Debt Process III, because of the correlation, it is more likely that offsetting debts are generated intertemporally. As a result, it is more cost efficient to not clear too frequently and reserve some debts in the system to offset future debts in opposite directions. This explains why the cost gap between the optimal policy and Policy 1 widens in Debt Process III. The cost of Policy 2 depends only on the unit liquidity cost and the total debt in the system. Overall we see that the heuristic Policy 3, based upon Proposition 3.7 is performing extremely well, as its cost is close to that of the optimal policy. In many settings, the heuristic policy is to clear all debts in every period and thus it is equivalent to Policy 1. When it is optimal to clear less frequently, e.g., the set-up cost increases, the heuristic Policy 3 effectively deviates from Policy 1 to reduce cost.

Table 6 depicts the average and 95% confidence interval of three non-cost measures under the clearing policies. We remind that for Policy 1, these measures are always zero and we thus do not report them in the table. Note that the non-cost measures depend on the cost parameters and the debt processes indirectly through the clearing policy. Therefore, as Policy 2 is independent of the cost parameters, so are these non-cost measures. Under all cost parameters, the average clearing time of the optimal policy is below one period, significantly lower than that of Policy 2. The maximum debt level under the optimal policy is less than 5, which is 25 times the standard deviation of the pairwise debt in each period. Note that even though the mean debt in each period is zero (Debt Process I), i.e., there is no systematic debt generated between banks, the maximum debt can be as high as 47 under Policy 2, which is more than 8 times higher than the optimal policy. For the average backlog of Bank 1, it is larger under Debt Process II but still manageable under the optimal policy: the average new debt in each period owed by Bank 1 is 0.8 and the average backlog is less than that in all scenarios. The performance of Policy 3 is very close to that of the optimal policy, in terms of cost and non-cost measures. For example, when $\delta = M = H = 0$ in the optimal policy, they are also zero in Policy 3, implying that both policies clear in every period in these scenarios. In some settings, Policy 3 even outperforms the optimal policy in terms of non-cost measures. For example, in the setting $K = 8$ and $\beta_l = 1$ of Debt Process III, Policy 3 is to clear every two periods for the first two clearings and every three periods for the last four clearings. All the non-cost measures under Policy 3 are about half of those under the optimal policy. But the cost gap of Policy 3 and the optimal policy is also largest in this setting, implying that Policy 3 clears too frequently and potentially incurs a large set-up cost.

Process	Parameter		Optimal			Policy 1			Policy 2			Policy 3		
	K	β_l	V	L	C	V	L	C	V	L	C	V	L	C
I	0	1	6.8 ± 0.4	0	6.8 ± 0.4	7.2 ± 0.4	0	7.2 ± 0.4	384.8 ± 4.5	384.8 ± 4.5	0	7.2 ± 0.4	0	7.2 ± 0.4
	0	2	6.8 ± 0.4	0	6.8 ± 0.4	7.2 ± 0.4	0	7.2 ± 0.4	769.6 ± 8.9	769.6 ± 8.9	0	7.2 ± 0.4	0	7.2 ± 0.4
	4	1	58.5 ± 0.5	19.4 ± 0.6	39.5 ± 1.0	71.2 ± 0.4	0	71.2 ± 0.4	384.8 ± 4.5	384.8 ± 4.5	0	59.2 ± 0.6	25.6 ± 0.4	33.6 ± 0.4
	4	2	66.7 ± 0.4	1.2 ± 0.4	65.6 ± 0.6	71.2 ± 0.4	0	71.2 ± 0.4	769.6 ± 8.9	769.6 ± 8.9	0	71.2 ± 0.4	0	71.2 ± 0.4
	8	1	86.7 ± 0.6	30.2 ± 0.7	56.4 ± 1.0	135.2 ± 0.4	0	135.2 ± 0.4	384.8 ± 4.5	384.8 ± 4.5	0	87.2 ± 0.6	25.6 ± 0.4	61.6 ± 0.4
	8	2	111.7 ± 0.7	38.6 ± 1.2	73.0 ± 1.7	135.2 ± 0.4	0	135.2 ± 0.4	769.6 ± 8.9	769.6 ± 8.9	0	112.8 ± 0.9	51.2 ± 0.7	61.6 ± 0.4
II	0	1	14.9 ± 0.3	0	14.9 ± 0.3	15.9 ± 0.3	0	15.9 ± 0.3	475.1 ± 4.3	475.1 ± 4.3	0	15.9 ± 0.3	0	15.9 ± 0.3
	0	2	14.9 ± 0.3	0	14.9 ± 0.3	15.9 ± 0.3	0	15.9 ± 0.3	950.1 ± 8.5	950.1 ± 8.5	0	15.9 ± 0.3	0	15.9 ± 0.3
	4	1	71.3 ± 0.4	14.6 ± 0.8	56.7 ± 1.0	79.9 ± 0.3	0	79.9 ± 0.3	475.1 ± 4.3	475.1 ± 4.3	0	73.9 ± 0.3	3.9 ± 0.1	69.9 ± 0.3
	4	2	74.9 ± 0.3	0	74.9 ± 0.3	79.9 ± 0.3	0	79.9 ± 0.3	950.1 ± 8.5	950.1 ± 8.5	0	79.9 ± 0.3	0	79.9 ± 0.3
	8	1	100.9 ± 0.5	31.6 ± 0.4	69.3 ± 0.3	143.9 ± 0.3	0	143.9 ± 0.3	475.1 ± 4.3	475.1 ± 4.3	0	100.9 ± 0.5	31.6 ± 0.3	69.3 ± 0.3
	8	2	129.0 ± 0.6	23.3 ± 1.5	105.7 ± 1.9	143.9 ± 0.3	0	143.9 ± 0.3	950.1 ± 8.5	950.1 ± 8.5	0	133.8 ± 0.4	7.9 ± 0.2	125.9 ± 0.3
III	0	1	6.2 ± 0.4	0	6.2 ± 0.4	6.7 ± 0.4	0	6.7 ± 0.4	190.1 ± 4.1	190.1 ± 4.1	0	6.7 ± 0.4	0	6.7 ± 0.4
	0	2	6.2 ± 0.4	0	6.2 ± 0.4	6.7 ± 0.4	0	6.7 ± 0.4	380.1 ± 8.1	380.1 ± 8.1	0	6.7 ± 0.4	0	6.7 ± 0.4
	4	1	45.0 ± 0.6	15.2 ± 0.4	29.8 ± 0.7	70.0 ± 0.6	0	70.0 ± 0.6	190.1 ± 4.1	190.1 ± 4.1	0	46.2 ± 0.5	13.0 ± 0.3	33.3 ± 0.4
	4	2	56.0 ± 0.7	10.2 ± 0.6	45.8 ± 1.1	70.0 ± 0.6	0	70.0 ± 0.6	380.1 ± 8.1	380.1 ± 8.1	0	70.0 ± 0.6	0	70.0 ± 0.6
	8	1	63.5 ± 0.7	28.1 ± 0.6	35.5 ± 0.8	133.3 ± 0.9	0	133.3 ± 0.9	190.1 ± 4.1	190.1 ± 4.1	0	74.2 ± 0.5	13.0 ± 0.3	61.3 ± 0.4
	8	2	84.8 ± 1.0	29.5 ± 0.7	55.4 ± 1.2	133.3 ± 0.9	0	133.3 ± 0.9	380.1 ± 8.1	380.1 ± 8.1	0	87.2 ± 0.8	25.9 ± 0.7	61.3 ± 0.4

Table 5: Policy comparison in terms of the total cost V, the liquidity cost L and the clearing cost C. The 95% confidence interval is illustrated after $\pm$.

Process	Parameter		Optimal			Policy 2			Policy 3			
	K	β_l	δ	M	H	δ	M	H	$\tau \times c$	δ	M	H
I	0	1	0	0	0	7.52 ± 0.04	47.80 ± 0.42	4.87 ± 0.10	1×16	0	0	0
	0	2	0	0	0				1×16	0	0	0
	4	1	0.39 ± 0.01	3.47 ± 0.06	0.24 ± 0.01				2×8	0.50 ± 0.00	3.96 ± 0.07	0.32 ± 0.01
	4	2	0.01 ± 0.01	0.61 ± 0.20	0.01 ± 0.00				1×16	0	0	0
	8	1	0.60 ± 0.02	5.22 ± 0.22	0.38 ± 0.01				2×8	0.50 ± 0.00	3.96 ± 0.07	0.32 ± 0.01
	8	2	0.39 ± 0.01	3.45 ± 0.05	0.24 ± 0.01				2×8	0.50 ± 0.00	3.96 ± 0.07	0.32 ± 0.01
II	0	1	0	0	0	7.48 ± 0.03	59.33 ± 0.44	10.74 ± 0.10	1×16	0	0	0
	0	2	0	0	0				1×16	0	0	0
	4	1	0.24 ± 0.01	4.07 ± 0.07	0.36 ± 0.02				$1 \times 14 + 2 \times 1$	0.06 ± 0.00	3.92 ± 0.11	0.09 ± 0.00
	4	2	0	0	0				1×16	0	0	0
	8	1	0.50 ± 0.00	4.76 ± 0.07	0.71 ± 0.01				2×8	0.50 ± 0.00	4.76 ± 0.07	0.71 ± 0.01
	8	2	0.19 ± 0.01	3.92 ± 0.08	0.29 ± 0.02				$1 \times 14 + 2 \times 1$	0.06 ± 0.00	3.92 ± 0.11	0.09 ± 0.00
III	0	1	0.00 ± 0.00	0.05 ± 0.03	0.00 ± 0.00	7.48 ± 0.07	24.14 ± 0.38	2.41 ± 0.11	1×16	0	0	0
	0	2	0.00 ± 0.00	0.05 ± 0.03	0.00 ± 0.00				1×16	0	0	0
	4	1	0.61 ± 0.02	2.89 ± 0.13	0.20 ± 0.01				2×8	0.51 ± 0.01	2.41 ± 0.07	0.16 ± 0.01
	4	2	0.23 ± 0.02	1.54 ± 0.06	0.07 ± 0.01				1×16	0	0	0
	8	1	1.15 ± 0.03	4.55 ± 0.12	0.36 ± 0.02				$2 \times 2 + 3 \times 4$	0.51 ± 0.01	2.41 ± 0.07	0.16 ± 0.01
	8	2	0.59 ± 0.02	2.81 ± 0.12	0.19 ± 0.01				2×8	0.51 ± 0.01	2.41 ± 0.07	0.16 ± 0.01

Table 6: Policy comparison in terms of the average clearing time δ, the maximum debt level M and the average backlog of Bank 1 H. The 95% confidence interval is illustrated after $\pm$. Policy 3 is specified in terms of cycle length τ multiplying clearings c.

In terms of the computation time, Policies 1 to 3 can be computed very efficiently because it doesn't scale with the dimension of the state. For the optimal policy, the computation time is around a minute, thanks to the reduction of the state space we discuss in Section 3. For real-world financial systems, there are typically more banks (there are 16 in the case study in the next section) and time periods. In this case, the computation time scales up exponentially with the number of banks and linearly with the number of periods for the optimal policy. Policy 3 based on the deterministic approximation would be favorable due to its robust performance and efficient computation.

4.2 Case Study of Payments Canada

In this section, we conduct a case study using the data obtained from Payments Canada. Payments Canada is a public purpose, non-profit organization that owns and operats Canada's payment clearing and settlement infrastructure. Payments Canada has been operating LVTS in the clearing and settlement of electronic wire payments among its members since 1999. The LVTS is a real-time net settlement system in which payment obligations between members are updated in real-time with final multilateral net debit positions settled on the books of the Bank of Canada at the end of each day. For detailed information of the LVTS, please refer to Arjani and McVanel (2006). The LVTS has a hybrid settlement mechanism where Tranche 1 payments are settled similarly to a real-time gross settlement system and Tranche 2 payments settled in an LSM-type fashion. Note that in this case study, we use clearing or settlement interchangeably.

As of the time of the analysis, Payments Canada had 16 member banks, whose net debit positions during a specific day were shown in Table 7 (for confidentiality we focus on a single, unspecified, day). With an increasing demand for real-time settlement, in September 2021 Payments Canada switched to a new infrastructure that allows it to settle the debts in the system during the day. Specifically, it replaced LVTS with Lynx that can better process time-critical payments and adapt to a dynamic payments environment. Below we conduct a preliminary analysis regarding the optimal clearing policy under different cost settings.

Net debt								Total debt
Bank 1	-5.224	Bank 5	0.068	Bank 9	-22.675	Bank 13	-15.921	461.816
Bank 2	274.881	Bank 6	-0.010	Bank 10	-1.034	Bank 14	-75.671	
Bank 3	-0.698	Bank 7	-33.964	Bank 11	-14.203	Bank 15	-3.383	
Bank 4	-19.440	Bank 8	-9.646	Bank 12	-0.002	Bank 16	-73.079	

Table 7: Debts (in billion dollars) incurred in LVTS during a typical day.

To provide more background information for the LVTS operation, the system runs for 18 hours a day. We divide the day into 5-minute intervals, i.e., periods in our analysis. The infrastructure does not record the distribution of the intraday debts generated between member banks. Therefore, we use the deterministic process (Section 3.3) to model the debt. In particular, we assume the debt in Table 7 is generated evenly and deterministically among the 216 total periods. To obtain some rough estimates, we first provide a calibration for β_c, the coordination cost, which can be determined by the effective daily rate for overdrafts. Specifically, the clearing of net debit positions typically requires daylight overdrafts which charge an effective daily rate. Following the overdraft policy of the Federal Reserve (Afonso & Shin, 2011), we calculate the effective daily rate denoted by R through

$$R = \bar{R} \times (h/24) \times (1/360)$$

where $\bar{R}$ is the annual rate charged for overdrafts, and h is the operating hours of the payment system (18 in our case). We use the target overnight rate set by the Bank of Canada to approximate $\bar{R}$, i.e., letting $\bar{R} = 0.0025$. Then, $R = 5.208 \times 10^{-6}$. By taking the average over the 216 periods, we get the period rate, i.e., the unit coordination cost $\beta_c = R/217 = 2.4 \times 10^{-8}$. As highlighted before Proposition 3.5, given the development of central banks' overdraft facility and the trend of real-time clearing, it is sensible to assume that $\beta_l \geq \beta_c$ (required in Proposition 3.7). For ease of analysis, we assume $\beta_l = \beta_c$. We also vary K to inspect the clearing policy accordingly.

Figure 2 demonstrates the optimal number of periods in a clearing cycle as a function of the set-up cost, K, in Canadian dollars. We observe that the optimal number of periods per clearing cycle is non-decreasing with the set-up cost. Specifically, when $K \leq \$51$ (the set-up cost is less than \$51), it is optimal to clear every period (i.e., every five minutes), except possibly for the last few periods. (Note that by Proposition 3.7, clearing cycles of length two periods may also be present.) When $\$52 \leq K \leq \$153$, the optimal clearing cycles are mainly of length two periods. This interval can be further divided, which gives rise to two cases: when $\$52 \leq K \leq \$72$, it may be optimal to clear every period for the first clearing and every two periods for the remaining periods; when

$\$73 \leq K \leq \$153$, it may be optimal to clear every two periods at first and every three periods for the last clearing. When $\$154 \leq K \leq \$307$, the optimal clearing cycles are mainly of length three. When $K \geq \$610,000$, it is optimal to clear only at the end of the day, which is the current practice. Based on this preliminary analysis, the real-time clearing regime would significantly reduce the cost for Payments Canada as long as the set-up cost is not very high.

To understand the clearing frequency in the global context, many countries and areas such as Eurosystem and U.K. are adopting a policy that clears the debt in every period. Their set-up cost is likely to be low. Korea's payment system, BOK-Wire+, implements the multilateral offsetting every 30 minutes. That is equivalent to clearing every six periods in our setting. If their liquidity and coordination costs are similar to the ones in our study, this clearing policy is optimal if its set-up cost is in the range from \$791 to \$1,077. As another example, Japan's payment system, BOJ-NET, implements the multilateral offsetting four times a day, i.e. clearing every fifty-four periods. In similar settings, this will be optimal if its set-up cost is in the range of \$61,000 to \$100,000.

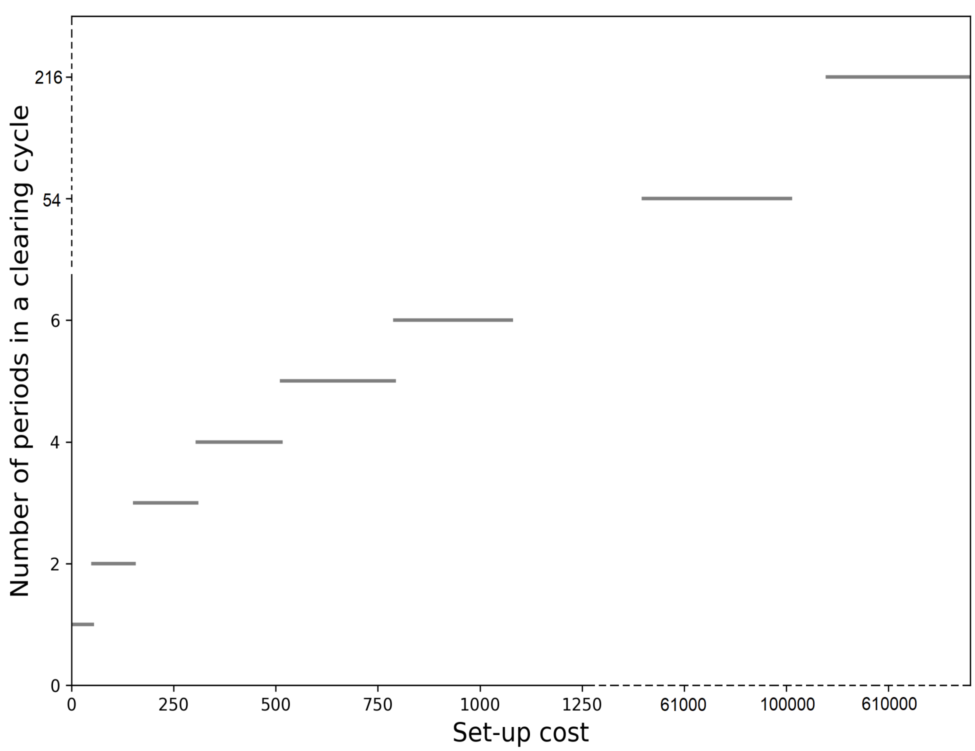

Figure 2: Optimal number of periods in a clearing cycle varying with the set-up cost for a typical case of Payments Canada.

5 Conclusion

In this paper, we provide a model for the interbank payment dynamics and use it to study the optimal clearing policy of the central bank. We apply state space reduction to approach the dynamic programming to make it tractable. As a result, we discover a surprising structure of the optimal clearing policy: either to clear all the debts in the system or not to clear at all in each period. Based on this structure, we study the deterministic approximation of the problem and conduct a comprehensive numerical study and a short case study.

There are several interesting directions for future research. In practice, there are other types of constraints the central bank may impose for various reasons. For example, there may be a net debit cap for daily overdraft or intraday credit limit for clearing net debts. In this case, the dynamic programming needs a state to record the remaining credit the central bank can lend for the rest of the horizon. Another practical consideration is to incorporate richer cost structures. For example, according to the studies on the investment of liquidity management technology (Heller & Lengwiler, 2003; Maddaloni, 2015), the coordination cost may be highly non-linear: it is relatively easy to coordinate banks and clear net debt below certain threshold but becomes significantly costly when the net debt is above the threshold. The liquidity cost may also be piecewise: it imposes any little risk to the system until the total debt hits a regulatory threshold. It remains an open problem to investigate the structure of the optimal policy under these extensions.

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A Proofs

Proof of Proposition 3.1: For $t = 0$, since $\Pi_0(\boldsymbol{B}) \equiv 0$, we simply set $V_0 \equiv 0$ and the statement holds. To show the claim for $t > 0$, by the Bellman equation (2), it is sufficient to show that for $\boldsymbol{B}_t$ and $\boldsymbol{B}'_t$ such that $D_t = \sum_{i,j} b_{ij,t} = \sum_{i,j} b'_{ij,t} = D'_t$ and $b_{i,t} = \sum_{j=1}^{N}(b_{ij,t} - b_{ji,t}) = \sum_{j=1}^{N}(b'_{ij,t} - b'_{ji,t}) = b'_{i,t}$ for all i, we have

$$\begin{aligned}
&\min_{\tilde{\boldsymbol{B}}_t} \left\{ K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l \tilde{D}_t + \mathbb{E}\left[\Pi_{t-1}(\tilde{\boldsymbol{B}}_t + \boldsymbol{X}_t)\right]\right\} \\
=&\min_{\tilde{\boldsymbol{B}}_t} \left\{ K\mathbb{1}_{D'_t \neq \tilde{D}_t} + \beta_c(d'_t - \tilde{d}_t) + \beta_l \tilde{D}_t + \mathbb{E}\left[\Pi_{t-1}(\tilde{\boldsymbol{B}}_t + \boldsymbol{X}_t)\right]\right\}.
\end{aligned}$$

Note that by the condition of $\boldsymbol{B}_t$ and $\boldsymbol{B}'_t$, we have $D_t = D'_t$; and because $\boldsymbol{b}_t = \boldsymbol{b}'_t$, we have $d_t = d'_t$. As a result, the same $\tilde{\boldsymbol{B}}_t$ would be feasible for both Bellman equations and lead to the same objective value. Therefore, the claim holds for all $t > 0$. □

Proof of Proposition 3.2: For $t = 0$, $V_t = \Pi_t \equiv 0$ and the claim holds automatically. For $t > 0$, suppose $\tilde{\boldsymbol{B}}^*_t$ is the optimal solution to (2). Let $\tilde{b}_{i,t} = \sum_{j=1}^{N}(\tilde{b}^*_{ij,t} - \tilde{b}^*_{ji,t})$ and $\tilde{D}_t = \sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{b}^*_{ij,t}$. We first show that $\tilde{\boldsymbol{b}}_t$ and $\tilde{D}_t$ satisfy the constraints in (3). Because the constraints in (2) satisfied by $\tilde{\boldsymbol{B}}^*_t$ are largely consistent with (3), We only need to establish the extra one, $\tilde{D}_t \geq \tilde{d}_t$, is also satisfied by $\tilde{\boldsymbol{B}}^*_t$. Note that

$$\begin{aligned}
\tilde{D}_t &= \sum_{i=1}^{N}\sum_{j=1}^{N}\tilde{b}^*_{ij,t} = \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}(\tilde{b}^*_{ij,t} + \tilde{b}^*_{ji,t}) \geq \sum_{i=1}^{N}\sum_{j=1}^{N}\frac{1}{2}|\tilde{b}^*_{ij,t} - \tilde{b}^*_{ji,t}| \\
&\geq \sum_{i=1}^{N}\frac{1}{2}\left|\sum_{j=1}^{N}(\tilde{b}^*_{ij,t} - \tilde{b}^*_{ji,t})\right| = \sum_{i=1}^{N}\frac{1}{2}\left|\tilde{b}_{i,t}\right| = \tilde{d}_t.
\end{aligned}$$

Here the first inequality is because $\tilde{b}^*_{ij,t} \geq 0$ by the constraint of (2). The last equality is due to $\sum_{i=1}^{N}\tilde{b}_{i,t} = 0$. Now that $\tilde{\boldsymbol{b}}_t$ and $\tilde{D}_t$ are feasible for (3) and $\mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})] = \mathbb{E}[\Pi_{t-1}(\boldsymbol{B}_{t-1})]$ by the definition of V_{t-1}, it proves that

$$\begin{aligned}
V_t(\boldsymbol{b}_t, D_t) = \Pi_t(\boldsymbol{B}_t) &= \min_{\tilde{\boldsymbol{B}}_t}\left\{K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l\tilde{D}_t + \mathbb{E}\left[\Pi_{t-1}(\tilde{\boldsymbol{B}}_t + \boldsymbol{b}_t)\right]\right\} \\
&= K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l\tilde{D}_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})\right] \\
&\geq \min_{\tilde{D}_t, \tilde{\boldsymbol{b}}_t}\left\{K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l\tilde{D}_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})\right]\right\}.
\end{aligned}$$

To show the opposite direction, i.e.,

$$V_t(\boldsymbol{b}_t, D_t) \le \min_{\tilde{D}_t, \tilde{\boldsymbol{b}}_t} \left\{ K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l \tilde{D}_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})\right] \right\},$$

subject to the constraints in (3), suppose $\tilde{\boldsymbol{b}}_t^*$ and D_t^* is an optimal solution to (3). We construct $\tilde{\boldsymbol{B}}_t$ using Algorithm 1. It is clear from the construction that the constraints in (2) are satisfied: $\sum_{j=1}^N (\tilde{b}_{ij,t} - \tilde{b}_{ji,t}) = \tilde{b}_{i,t}^*$ and $\sum_{i=1}^N \sum_{j=1}^N \tilde{b}_{ij,t} = \tilde{D}_t^*$. Plugging this $\tilde{\boldsymbol{B}}_t$ into (2) implies

$$\begin{aligned} V_t(\boldsymbol{b}_t, D_t) = \Pi_t(\boldsymbol{B}_t) &\le K\mathbb{1}_{D_t \neq \tilde{D}_t^*} + \beta_c(d_t - \tilde{d}_t^*) + \beta_l \tilde{D}_t^* + \mathbb{E}\left[\Pi_{t-1}(\tilde{\boldsymbol{B}}_t + \boldsymbol{b}_t)\right] \\ &= \min_{\tilde{D}_t, \tilde{\boldsymbol{b}}_t} \left\{ K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l \tilde{D}_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})\right] \right\}. \end{aligned}$$

This proves the claim. □

Proof of Lemma 3.3: We first prove part (a) by induction. For $t = 0$, the claim holds because $V_0 \equiv 0$. Suppose the claim holds for period $t-1$. For the ease of analysis, we first partition the set $\{1, 2, \ldots, N\}$ into seven index sets:

$$\begin{aligned} &I_0 = \{i : b_{i,t} = b'_{i,t} = 0\}, \quad I_1 = \{i : b_{i,t} > b'_{i,t} \ge 0\}, \quad I_2 = \{i : b'_{i,t} \ge b_{i,t} \ge 0\} \setminus I_0, \\ &I_3 = \{i : b'_{i,t} > 0 > b_{i,t}\}, \quad I_4 = \{i : b_{i,t} < b'_{i,t} \le 0\}, \quad I_5 = \{i : b'_{i,t} \le b_{i,t} \le 0\} \setminus I_0, \\ &I_6 = \{i : b_{i,t} > 0 > b'_{i,t}\}. \end{aligned} \tag{7}$$

We are going to show $V_t(\boldsymbol{b}'_t, D_t) - V_t(\boldsymbol{b}_t, D_t) \le \beta_c u$ for two separate cases. The first case is $d'_t < d_t$ and the second case is $d'_t \ge d_t$. In the first case, suppose $(\tilde{\boldsymbol{b}}_t^*, \tilde{D}_t^*)$ is the optimal solution to $V_t(\boldsymbol{b}_t, D_t)$. We are going to construct a feasible solution $(\tilde{\boldsymbol{b}}'_t, \tilde{D}_t^*)$ to $V_t(\boldsymbol{b}'_t, D_t)$ satisfying the following conditions:

$$\sum_{i=1}^N (\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+ \le u, \tag{8a}$$

$$d'_t - \tilde{d}'_t \le d_t - \tilde{d}_t^*. \tag{8b}$$

If such solution exists, then we have

$$\begin{aligned} V_t(\boldsymbol{b}'_t, D_t) - V_t(\boldsymbol{b}_t, D_t) &\le K\mathbb{1}_{D_t \neq \tilde{D}_t^*} + \beta_c(d'_t - \tilde{d}'_t) + \beta_l \tilde{D}_t^* + \mathbb{E}\left[V_{t-1}(\tilde{\boldsymbol{b}}'_t + \boldsymbol{X}_t, D_{t-1})\right] \\ &\quad - \left[K\mathbb{1}_{D_t \neq \tilde{D}_t^*} + \beta_c(d_t - \tilde{d}_t^*) + \beta_l \tilde{D}_t^* + \mathbb{E}\left[V_{t-1}(\tilde{\boldsymbol{b}}_t^* + \boldsymbol{X}_t, D_{t-1})\right]\right] \\ &= \beta_c(d'_t - \tilde{d}'_t) - \beta_c(d_t - \tilde{d}_t^*) + \mathbb{E}[V_{t-1}(\boldsymbol{b}'_{t-1}, D_{t-1})] - \mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})] \\ &\le \mathbb{E}[V_{t-1}(\boldsymbol{b}'_{t-1}, D_{t-1})] - \mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})] \\ &\le \beta_c u \end{aligned}$$

Condition	Scenarios	Projection	Effect
$b'_{i,t} \geq 0 > \tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t}$	I_1	$\tilde{b}'_{i,t} = 0$	$0 \geq \tilde{b}'_{i,t} - \tilde{b}^*_{i,t} \geq b'_{i,t} - b_{i,t}$
$b'_{i,t} > \tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t} \geq 0$	I_1, I_2	$\tilde{b}'_{i,t} = \tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t}$	$\tilde{b}'_{i,t} - \tilde{b}^*_{i,t} = b'_{i,t} - b_{i,t}$
$\tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t} \geq b'_{i,t} > 0$	I_3	$\tilde{b}'_{i,t} = b'_{i,t}$	$0 \leq \tilde{b}'_{i,t} - \tilde{b}^*_{i,t} \leq b'_{i,t} - b_{i,t}$
$b'_{i,t} \leq 0 < \tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t}$	I_4	$\tilde{b}'_{i,t} = 0$	$0 \leq \tilde{b}'_{i,t} - \tilde{b}^*_{i,t} \leq b'_{i,t} - b_{i,t}$
$b'_{i,t} < \tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t} \leq 0$	I_4, I_5	$\tilde{b}'_{i,t} = \tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t}$	$\tilde{b}'_{i,t} - \tilde{b}^*_{i,t} = b'_{i,t} - b_{i,t}$
$\tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t} \leq b'_{i,t} < 0$	I_6	$\tilde{b}'_{i,t} = b'_{i,t}$	$0 \geq \tilde{b}'_{i,t} - \tilde{b}^*_{i,t} \geq b'_{i,t} - b_{i,t}$

Table 8: The effect of the projection is based on different scenarios in the first stage of the first case. We omit I_0 which is a trivial case.

The last inequality follows from $\sum_{i=1}^N (b'_{i,t-1} - b_{i,t-1})^+ = \sum_{i=1}^N (\tilde{b}'_{i,t} + \sum_{j=1}^N (X_{ij} - X_{ji}) - \tilde{b}^*_{i,t} - \sum_{j=1}^N (X_{ij} - X_{ji}))^+ = \sum_{i=1}^N (\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+ \leq u$ (due to (1)) and the inductive hypothesis. This proves the claim for t.

It remains to construct a feasible solution that satisfies (8a) and (8b). We construct $\tilde{\boldsymbol{b}}'_t$ from $\tilde{\boldsymbol{b}}^*_t$ in three stages. In the first stage, we shift $\tilde{\boldsymbol{b}}^*_t$ by $\boldsymbol{b}'_t - \boldsymbol{b}_t$ and project it to $[0, \boldsymbol{b}'_t]$: let $\tilde{b}'_{i,t} \leftarrow \mathrm{Proj}_{[0, b'_{i,t}]}(\tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t})$ for $i = 1, \ldots, N$. Note that $b'_{i,t}$ may be negative, in which case the projected interval is $[b'_{i,t}, 0]$. Because of the projection, the conditions $|\tilde{b}'_{i,t}| \leq |b'_{i,t}|$ and $b'_{i,t}\tilde{b}'_{i,t} \geq 0$ are easily shown to be satisfied, which is the feasibility condition (3c). Moreover, since $|\tilde{b}^*_{i,t}| \leq |b_{i,t}|$ and $b_{i,t}\tilde{b}^*_{i,t} \geq 0$, the possible scenarios according to (7) and the effect of the projection operation are shown in Table 8. Verifying all the scenarios, it can be shown that the projection leads to $(\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+ \leq (b'_{i,t} - b_{i,t})^+$ and $(\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^- \leq (b'_{i,t} - b_{i,t})^-$, which implies (8a). As a result, after the first stage, we have

$$(\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+ \leq (b'_{i,t} - b_{i,t})^+, \tag{9a}$$

$$(\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^- \leq (b'_{i,t} - b_{i,t})^-, \tag{9b}$$

$$|\tilde{b}'_{i,t}| \leq |b'_{i,t}|, \quad b'_{i,t}\tilde{b}'_{i,t} \geq 0, \tag{9c}$$

for all $i = 1, \ldots, N$.

In the second stage, we transform the $\tilde{\boldsymbol{b}}'_t$ output from the first stage to guarantee $\sum_{i=1}^N \tilde{b}'_{i,t} = 0$ and $d'_t - \tilde{d}'_t \leq d_t - \tilde{d}^*_t$ while keeping the conditions (9) satisfied. If $\sum_{i=1}^N \tilde{b}'_{i,t} = 0$, then we are done. Otherwise, without loss of generality, suppose $\sum_{i=1}^N \tilde{b}'_{i,t} = \epsilon > 0$ after the first stage. Recall the index sets (7). We decrease the values of all $\tilde{b}'_{i,t}$ for $i \in I_1 \cup I_2 \cup I_3$ gradually toward zero. (Recall that the signs of $\tilde{b}$ and b are the same.) Note that in the process, (9a) and (9c) are always satisfied. Eventually, at some point, the total decreased amount reaches ϵ (because $\sum_{i=1}^N \tilde{b}'_{i,t} \geq \epsilon$) and as a result we have $\sum_{i=1}^N \tilde{b}'_{i,t} = 0$. Combined with the fact that $\sum_{i=1}^N \tilde{b}^*_{i,t} = 0$, it also implies that $\sum_{i=1}^N (\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^- = \sum_{i=1}^N (\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+ \leq u$. We note that with this definition, the $\epsilon < 0$ case is symmetric and we thus use the same method but keep (9b) hold instead of (9a).

Next we prove that the output of the second stage satisfies (8b), $d'_t - \tilde{d}'_t \leq d_t - \tilde{d}^*_t$. Note that $\tilde{b}'_{i,t}$ for $i \in I_4 \cup I_5 \cup I_6$ does not change in the second stage as $\epsilon \geq 0$. By the definition of the debt level (3b) and Table 8, we have

$$
\begin{aligned}
\tilde{d}'_t - \tilde{d}^*_t &= \sum_{i=1}^{N} (\tilde{b}'_{i,t})^- - \sum_{i=1}^{N} (\tilde{b}^*_{i,t})^- \\
&= -\sum_{i \in I_4 \cup I_5} \min\{\tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t}, 0\} - \sum_{i \in I_6} b'_{i,t} + \sum_{i \in I_4 \cup I_5} \tilde{b}^*_{i,t} + \sum_{i \in I_3} \tilde{b}^*_{i,t} \\
&\geq -\sum_{i \in I_4 \cup I_5} (\tilde{b}^*_{i,t} + b'_{i,t} - b_{i,t}) - \sum_{i \in I_6} b'_{i,t} + \sum_{i \in I_4 \cup I_5} \tilde{b}^*_{i,t} + \sum_{i \in I_3} b_{i,t} \\
&= -\sum_{i \in I_4 \cup I_5} (b'_{i,t} - b_{i,t}) - \sum_{i \in I_6} b'_{i,t} + \sum_{i \in I_3} b_{i,t} \\
&= d'_t - d_t. \qquad (10)
\end{aligned}
$$

At this point, we have shown that (8), (3a) and the first two constraints of (3c) are satisfied.

If $\tilde{d}'_t \leq \tilde{d}^*_t$, it follows directly that $\tilde{d}'_t \leq \tilde{d}^*_t \leq \tilde{D}^*_t$ and the last constraint of (3c) holds. In this case, the constructed $(\tilde{\boldsymbol{b}}'_t, \tilde{D}^*_t)$ is feasible and satisfies (8).

If $\tilde{d}'_t > \tilde{d}^*_t$, then we add a third stage to transform the output of the second stage to satisfy $\tilde{d}'_t \leq \tilde{D}^*_t$ while keeping other constraints satisfied. We gradually decrease $\tilde{b}'_{i,t}$ toward $\tilde{b}^*_{i,t}$ if $\tilde{b}'_{i,t} \geq \tilde{b}^*_{i,t} \geq 0$; and decrease $\tilde{b}'_{i,t}$ toward 0 if $\tilde{b}'_{i,t} \geq 0 \geq \tilde{b}^*_{i,t}$. Note that this case differs from under I_i because the order of debt in $\tilde{b}$ may differ from that in b. Meanwhile, symmetrically, we gradually increase $\tilde{b}'_{i,t}$ toward $\tilde{b}^*_{i,t}$ if $\tilde{b}'_{i,t} \leq \tilde{b}^*_{i,t} \leq 0$; increase $\tilde{b}'_{i,t}$ toward 0 if $\tilde{b}'_{i,t} \leq 0 \leq \tilde{b}^*_{i,t}$. We keep the sum $\sum_{i=1}^{N} \tilde{b}'_{i,t} = 0$ unchanged. In the process, as $\tilde{d}'_t$ is decreasing continuously, at some point, we must have $\tilde{d}'_t = \tilde{d}^*_t$. (Otherwise, suppose all the decreasing $\tilde{b}'_{i,t}$ reaches $\max\{\tilde{b}^*_{i,t}, 0\}$ and we still have $\tilde{d}'_t > \tilde{d}^*_t$. This results in a contradiction because for all $\tilde{b}'_{i,t} > 0$, we have $\tilde{b}'_{i,t} \leq \tilde{b}^*_{i,t}$ after the decrease and thus $\tilde{d}'_t = \sum_{i=1}^{N} (\tilde{b}'_{i,t})^+ \leq \sum_{i=1}^{N} (\tilde{b}^*_{i,t})^+ = \tilde{d}^*_t$.) Now consider the new $\tilde{\boldsymbol{b}}'_t$ with $\tilde{d}'_t = \tilde{d}^*_t \leq \tilde{D}^*_t$. The decreasing apparently doesn't change other feasibility conditions. For (8a), it still holds because the decrease only makes $\sum_{i=1}^{N} (\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+$ smaller. For (8b), note that by (10) we now have $\tilde{d}'_t = \tilde{d}^*_t$ and $d'_t < d^*_t$ (the first case). Therefore, all feasibility conditions and (8) hold and we have completed the proof for the first case.

Next, we investigate the second case $d'_t \geq d_t$. In this case, there may not exist a feasible solution $(\tilde{\boldsymbol{b}}'_t, \tilde{D}^*_t)$ satisfying (8), for example when $\tilde{d}^*_t = \tilde{D}^*_t$. (If such a feasible solution exists, the claims hold just as in the first case.) We are thus going to construct a feasible solution $(\tilde{\boldsymbol{b}}'_t, \tilde{D}^*_t)$ to

$V_t(\boldsymbol{b}'_t, D_t)$ satisfying the following conditions:

$$\sum_{i=1}^{N}(\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+ \le (1-w)u, \quad \tilde{d}'_t = \tilde{d}^*_t \tag{11}$$

where $w = \frac{d'_t - d_t}{u} \in (0,1)$ because $d'_t - d_t \le \sum_{i=1}^{N}(b'_{i,t} - b_{i,t})^+ \le u$ (recall that $x^+ - y^+ \le (x-y)^+$). If such solution exists, then we have

$$\begin{aligned}
V_t(\boldsymbol{b}'_t, D_t) - V_t(\boldsymbol{b}_t, D_t) &\le K\mathbb{1}_{D_t \ne \tilde{D}^*_t} + \beta_c(d'_t - \tilde{d}'_t) + \beta_l \tilde{D}^*_t + \mathbb{E}\left[V_{t-1}(\tilde{\boldsymbol{b}}'_t + \boldsymbol{X}_t, D_{t-1})\right] \\
&\quad - \left[K\mathbb{1}_{D_t \ne \tilde{D}^*_t} + \beta_c(d_t - \tilde{d}^*_t) + \beta_l \tilde{D}^*_t + \mathbb{E}\left[V_{t-1}(\tilde{\boldsymbol{b}}^*_t + \boldsymbol{X}_t, D_{t-1})\right]\right] \\
&= \beta_c(d'_t - d_t) + \mathbb{E}[V_{t-1}(\boldsymbol{b}'_{t-1}, D_{t-1})] - \mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})] \\
&\le \beta_c wu + \beta_c(1-w)u \\
&= \beta_c u
\end{aligned}$$

The last inequality follows from $\sum_{i=1}^{N}(b'_{i,t-1} - b_{i,t-1})^+ = \sum_{i=1}^{N}(\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+ \le (1-w)u$ and the inductive hypothesis.

It remains to construct a feasible solution that satisfies (11). Here we follow a different approach from the first case because the constraint (11) is different from (8). We first consider a new state variable $\boldsymbol{b}''_t$ such that

$$d''_t = d_t, \quad \sum_{i=1}^{N}(b''_{i,t} - b_{i,t})^+ \le (1-w)u, \quad |b''_{i,t}| \le |b'_{i,t}|, \quad b'_{i,t}b''_{i,t} \ge 0, \quad \sum_{i=1}^{N} b''_{i,t} = 0. \tag{12}$$

Such $\boldsymbol{b}''_t$ can be constructed from $\boldsymbol{b}'_t$ as follows. Recall the index sets (7). We gradually decrease $b'_{i,t}$ toward $b_{i,t}$ for $i \in I_2$ and decrease $b'_{i,t}$ toward 0 for $i \in I_3$. We refer to the new $b'_{i,t}$ as $b''_{i,t}$. In this process, we always decrease positive $b'_{i,t}$ so the third inequality of (12) holds. Meanwhile, symmetrically, we gradually increase $b'_{i,t}$ toward $b_{i,t}$ for $i \in I_5$ and increase $b'_{i,t}$ toward 0 for $i \in I_6$. Similarly, we always have the third inequality of (12) hold. We also keep the sum $\sum_{i=1}^{N} b''_{i,t} = 0$ unchanged. Note that this is the same process as in the third stage of the first case. In the process, as d''_t is decreasing continuously, at some point, we must have $d''_t = d_t$. (Otherwise, for example, suppose all the decreasing $b''_{i,t}$ reaches $\max\{b_{i,t}, 0\}$ and we still have $d''_t > d_t$. This results in a contradiction because for all $b''_{i,t} > 0$, we have $b''_{i,t} \le b_{i,t}$ after the decrease and thus $d''_t = \sum_{i=1}^{N}(b''_{i,t})^+ \le \sum_{i=1}^{N}(b_{i,t})^+ = d_t$.) We let the new stage after the decrease be $\boldsymbol{b}''_t$. Note that we must have

$$\sum_{i \in I_1 \cup I_2 \cup I_3} b'_{i,t} = d'_t = d_t + wu = d''_t + wu = \sum_{i \in I_1} b'_{i,t} + \sum_{i \in I_2 \cup I_3} b''_{i,t} + wu, \tag{13}$$

where the first equality follows from the definition of I_i in (7), the second follows from the definition of w in (11), and the last equality follows the construction of $b''_{i,t}$. Equation (13) implies that

$\sum_{i\in I_2\cup I_3}(b'_{i,t} - b''_{i,t}) = wu$. Thus, $\sum_{i=1}^N (b''_{i,t} - b_{i,t})^+ = \sum_{i\in I_2\cup I_3\cup I_4}(b''_{i,t} - b_{i,t}) = \sum_{i\in I_2\cup I_3\cup I_4}(b'_{i,t} - b_{i,t}) - wu = \sum_{i=1}^N (b'_{i,t} - b_{i,t})^+ - wu \le (1-w)u$, implying the second inequality in (12) and that all of the equations in (12) hold.

Note that any feasible solution to $V_t(\boldsymbol{b}''_t, D_t)$ is a feasible solution to $V_t(\boldsymbol{b}'_t, D_t)$. We consider $\boldsymbol{b}''_t$ as the new $\boldsymbol{b}'_t$ and construct $\tilde{\boldsymbol{b}}'_t$ from $\tilde{\boldsymbol{b}}^*_t$ through three stages. The first two stages are the same as the first case $d'_t < d_t$. After the second stage, we can get $\tilde{\boldsymbol{b}}'_t$ satisfying

$$d''_t - \tilde{d}'_t \le d_t - \tilde{d}^*_t, \tag{14a}$$

$$\sum_{i=1}^N (\tilde{b}'_{i,t} - \tilde{b}^*_{i,t})^+ \le (1-w)u, \tag{14b}$$

$$|\tilde{b}'_{i,t}| \le |b''_{i,t}|, \quad b''_{i,t}\tilde{b}'_{i,t} \ge 0, \quad \sum_{i=1}^N \tilde{b}'_{i,t} = 0, \tag{14c}$$

by the same argument as in the first case.

In the third stage, we transform $\tilde{\boldsymbol{b}}'_t$ to satisfy $\tilde{d}'_t = \tilde{d}^*_t$ while keeping (14b) and (14c) satisfied. From (14a), we have $\tilde{d}'_t - \tilde{d}^*_t \ge d'_t - d_t \ge 0$. Therefore, similar to the third stage in the first case, we can always transform $\tilde{\boldsymbol{b}}'_t$ to satisfy $\tilde{d}'_t = \tilde{d}^*_t$. As a result, $\tilde{d}'_t = \tilde{d}^*_t \le \tilde{D}^*_t$. Such $\tilde{\boldsymbol{b}}'_t$ is feasible to $V(\boldsymbol{b}''_t, D_t)$ and satisfies (11). This completes the second case and thus part (a).

We now prove part (b) by induction. For $t = 0$, both part (a) and (b) hold since $V_0 \equiv 0$. Suppose they hold for period $t-1$. We first establish the left-hand side. Suppose $D'_t \le D_t$. Let $(\tilde{\boldsymbol{b}}^*_t, \tilde{D}^*_t)$ be the optimal solution to $V_t(\boldsymbol{b}_t, D_t)$. Let $\tilde{D}'_t = \max\{D'_t - D_t + \tilde{D}^*_t, \tilde{d}^*_t\}$. It is easy to verify that $\tilde{d}^*_t \le \tilde{D}'_t \le D'_t$. Thus, $(\tilde{\boldsymbol{b}}^*_t, \tilde{D}'_t)$ is a feasible solution to $V_t(\boldsymbol{b}_t, D'_t)$. Moreover, we have $\tilde{D}'_t \le \tilde{D}^*_t$ because $D'_t \le D_t$ and $\tilde{d}^*_t \le \tilde{D}^*_t$. Then,

$$\begin{aligned} V_t(\boldsymbol{b}_t, D'_t) - V_t(\boldsymbol{b}_t, D_t) &\le K\mathbb{1}_{D'_t \ne \tilde{D}'_t} + \beta_c(d_t - \tilde{d}^*_t) + \beta_l \tilde{D}'_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, \tilde{D}'_t + X_t)\right] \\ &\quad - \left[K\mathbb{1}_{D_t \ne \tilde{D}^*_t} + \beta_c(d_t - \tilde{d}^*_t) + \beta_l \tilde{D}^*_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, \tilde{D}^*_t + X_t)\right]\right] \\ &\le \beta_l \tilde{D}'_t - \beta_l \tilde{D}^*_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D'_{t-1})] - \mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})] \\ &\le 0. \end{aligned}$$

The second inequality follows from the fact that $\mathbb{1}_{D'_t \ne \tilde{D}'_t} \le \mathbb{1}_{D'_t \ne \tilde{D}^*_t}$ because $D_t = \tilde{D}^*_t$ implies $\tilde{D}'_t = D'_t$. The last inequality follows from $D'_{t-1} = \tilde{D}'_t + X_t \le \tilde{D}^*_t + X_t = D_{t-1}$ and the inductive hypothesis. This proves the claim for t and thus completes the left-hand side.

We next establish the right-hand side. Suppose $D'_t \ge D_t$. Let $(\tilde{\boldsymbol{b}}^*_t, \tilde{D}^*_t)$ be the optimal solution to $V_t(\boldsymbol{b}_t, D_t)$. Let $\tilde{D}'_t = D'_t - D_t + \tilde{D}^*_t$. It is easy to verify that $\tilde{d}^*_t \le \tilde{D}'_t \le D'_t$. Thus, $(\tilde{\boldsymbol{b}}^*_t, \tilde{D}'_t)$ is a feasible solution to $V_t(\boldsymbol{b}_t, D'_t)$. Then,

$$\begin{aligned} V_t(\boldsymbol{b}_t, D'_t) - V_t(\boldsymbol{b}_t, D_t) &\le K\mathbb{1}_{D'_t \neq \tilde{D}'_t} + \beta_c(d_t - \tilde{d}^*_t) + \beta_l \tilde{D}'_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, \tilde{D}'_t + X_t)\right] \\ &\quad - \left[K\mathbb{1}_{D_t \neq \tilde{D}^*_t} + \beta_c(d_t - \tilde{d}^*_t) + \beta_l \tilde{D}^*_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, \tilde{D}^*_t + X_t)\right]\right] \\ &= \beta_l(D'_t - D_t) + \mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D'_{t-1})] - \mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})] \\ &\le t\beta_l(D'_t - D_t). \end{aligned}$$

The second inequality follows from the fact that $\mathbb{1}_{D'_t \neq \tilde{D}'_t} \le \mathbb{1}_{D'_t \neq \tilde{D}^*_t}$ because $D_t = \tilde{D}^*_t$ implies $\tilde{D}'_t = D'_t$. The last inequality follows from $D'_{t-1} = \tilde{D}'_t + X_t = D'_t - D_t + \tilde{D}^*_t + X_t = D'_t - D_t + D_{t-1}$ and the inductive hypothesis. □

Proof of Proposition 3.4: The proof consists of two steps. In the first step, we show that $\tilde{d}^*_t = \min\{d_t, \tilde{D}^*_t\}$, i.e., not clearing or clearing offsetting debt first. Consider any feasible solution $(\tilde{\boldsymbol{b}}_t, \tilde{D}_t)$ with $\tilde{d}_t < \min\{\tilde{D}_t, d_t\}$, not clearing or clearing offsetting debt first. Let $u = \min\{\tilde{D}_t, d_t\} - \tilde{d}_t > 0$. We are going to construct another feasible solution $(\tilde{\boldsymbol{b}}'_t, \tilde{D}_t)$ such that $\tilde{d}'_t = \min\{\tilde{D}'_t, d_t\}$ and

$$K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}_t) + \beta_l \tilde{D}_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})\right] > K\mathbb{1}_{D_t \neq \tilde{D}_t} + \beta_c(d_t - \tilde{d}'_t) + \beta_l \tilde{D}_t + \mathbb{E}\left[V_{t-1}(\boldsymbol{b}'_{t-1}, D_{t-1})\right], \tag{15}$$

which suffices to prove the claim by the Bellman equation.

Without loss of generality, suppose $\tilde{b}_{1,t} \le \tilde{b}_{2,t} \le \cdots \le \tilde{b}_{k,t} \le 0 \le \tilde{b}_{k+1,t} \le \cdots \le \tilde{b}_{N,t}$. By the constraint $\tilde{\boldsymbol{b}}_t \times \boldsymbol{b}_t \ge 0$, we have $b_{1,t}, b_{2,t}, \ldots, b_{k,t} \le 0$ and $b_{k+1,t}, \ldots, b_{N,t} \ge 0$ as well. Because $b_{i,t} \le \tilde{b}_{i,t} \le 0$ for $i = 1, \ldots, k$ and $\sum_{i=1}^k b_{i,t} = -d_t \le -\tilde{d}_t - u = \sum_{i=1}^k \tilde{b}_{i,t} - u$, we can always find $\tilde{b}'_{1,t}, \ldots, \tilde{b}'_{k,t}$ satisfying $b_{i,t} \le \tilde{b}'_{i,t} \le \tilde{b}_{i,t}$ for $i = 1, \ldots, k$ and $\sum_{i=1}^k \tilde{b}'_{i,t} = \sum_{i=1}^k \tilde{b}_{i,t} - u$. Similarly, we can find $\tilde{b}'_{k+1,t}, \ldots, \tilde{b}'_{N,t}$ satisfying $b_{i,t} \ge \tilde{b}'_{i,t} \ge \tilde{b}_{i,t}$ for $i = k+1, \ldots, N$ and $\sum_{i=k+1}^N \tilde{b}'_{i,t} = \sum_{i=k+1}^N \tilde{b}_{i,t} + u$.

It can be shown that such choice of $\tilde{\boldsymbol{b}}'_t$ satisfies the following conditions:

$$\begin{aligned} &\sum_{i=1}^N \tilde{b}'_{i,t} = \sum_{i=1}^k \tilde{b}'_{i,t} + \sum_{i=k+1}^N \tilde{b}'_{i,t} = \sum_{i=1}^k \tilde{b}_{i,t} - u + \sum_{i=k+1}^N \tilde{b}_{i,t} + u = 0 \\ &\tilde{d}'_t = \sum_{i=k+1}^N \tilde{b}'_{i,t} = \sum_{i=k+1}^N \tilde{b}_{i,t} + u = \min\{\tilde{D}_t, d_t\} \\ &|\tilde{b}'_{i,t} - \tilde{b}_{i,t}| \le u, \quad \forall i = 1, \ldots, N \\ &\sum_{i=1}^N (\tilde{b}'_{i,t} - \tilde{b}_{i,t})^+ = \sum_{i=k+1}^N (\tilde{b}'_{i,t} - \tilde{b}_{i,t}) = u. \end{aligned}$$

Therefore, the feasibility constraints in (3) are met.

Because of the last equality, by Lemma 3.3 part (a), we have $\mathbb{E}[V_{t-1}(\boldsymbol{b}'_{t-1}, D_{t-1})] - \mathbb{E}[V_{t-1}(\boldsymbol{b}_{t-1}, D_{t-1})] \le \beta_c u$. Plugging the inequality and $\tilde{d}'_t = u + \tilde{d}_t$ into (15), it is easy to verify that (15) indeed holds. Therefore, we have completed the first step.

In the second step, by (3) we have

$$V_t(\boldsymbol{b}_t, D_t) = \min_{\tilde{D}_t, \tilde{\boldsymbol{b}}_t} \left\{ K\mathbb{I}_{D_t \neq \tilde{D}_t} + U_t(\tilde{\boldsymbol{b}}_t, \tilde{D}_t | \boldsymbol{b}_t, D_t) \right\} \quad s.t. \quad \text{(3a)} - \text{(3d)} \tag{16}$$

where

$$U_t(\tilde{\boldsymbol{b}}_t, \tilde{D}_t | \boldsymbol{b}_t, D_t) = \begin{cases} \beta_c(d_t - \tilde{D}_t) + \beta_l \tilde{D}_t + \mathbb{E}[V_{t-1}(\tilde{\boldsymbol{b}}_t + \boldsymbol{X}_t, \tilde{D}_t + X_t)], & 0 \leq \tilde{D}_t \leq d_t \\ \beta_l \tilde{D}_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, \tilde{D}_t + X_t)], & d_t < \tilde{D}_t \leq D_t \end{cases}. \tag{17}$$

Here we have used the result of the first step: it is optimal to set $\tilde{d}_t = \min\{d_t, \tilde{D}_t\}$. Because V_{t-1} is increasing in $\tilde{D}_t + X_t$ (Lemma 3.3 part (b)) and thus $U_t(\tilde{\boldsymbol{b}}_t, \tilde{D}_t | \boldsymbol{b}_t, D_t)$ is strictly increasing in $\tilde{D}_t$ for $d_t < \tilde{D}_t \leq D_t$, we conclude that it is never optimal to let $\tilde{D}_t > d_t$ if $\tilde{D}_t \neq D_t$. This completes the proof. □

Proof of Proposition 3.5: We first prove that if it is optimal to clear, all debts must be cleared, i.e. $\tilde{\boldsymbol{b}}_t^* = \mathbf{0}, \tilde{D}_t^* = 0$. We will prove this claim by contradiction. Suppose it is optimal to clear in period t and moreover $\tilde{D}_t^* > 0$. By Proposition 3.4, we also have $\tilde{D}_t^* = \tilde{d}_t^* > 0$. By the definition (17) of the function U, we have

$$\begin{aligned} U_t(\tilde{\boldsymbol{b}}_t^*, \tilde{D}_t^* | \boldsymbol{b}_t, D_t) &= \beta_c(d_t - \tilde{D}_t^*) + \beta_l \tilde{D}_t^* + \mathbb{E}[V_{t-1}(\tilde{\boldsymbol{b}}_t^* + \boldsymbol{X}_t, \tilde{D}_t^* + X_t)] \\ &\geq \beta_c(d_t - \tilde{D}_t^*) + \beta_l \tilde{D}_t^* + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, \tilde{D}_t^* + X_t)] - \beta_c \tilde{D}_t^* \\ &\geq \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)] + (\beta_l - 2\beta_c)\tilde{D}_t^* \\ &\geq \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)] \\ &= U_t(\mathbf{0}, 0 | \boldsymbol{b}_t, D_t). \end{aligned}$$

The first inequality follows from $\sum_{i=1}^N (X_{i,t} - (\tilde{b}_{i,t}^* + X_{i,t}))^+ = \tilde{d}_t^* = \tilde{D}_t^*$ and Lemma 3.3 part (a). The second inequality holds because $V_t(\boldsymbol{b}_t, D_t)$ is increasing in D_t for any t (Lemma 3.3 part (b)). The third inequality follows from the condition that $\beta_l \geq 2\beta_c$. By the Bellman equation (16), it is optimal to use $(\mathbf{0}, 0)$ instead of $(\tilde{\boldsymbol{b}}_t^*, \tilde{D}_t^*)$, which leads to a contradiction.

We next characterize the region of the state $(\boldsymbol{b}_t, D_t)$ such that it is optimal to clear all debts. Based on the Bellman equation (16), it is optimal to clear if and only if $U_t(\boldsymbol{b}_t, D_t | \boldsymbol{b}_t, D_t) \geq K + U_t(\mathbf{0}, 0 | \boldsymbol{b}_t, D_t)$, i.e.,

$$\beta_l D_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, D_t + X_t)] \geq K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]. \tag{18}$$

We first show part one of the proposition. For any $\boldsymbol{b}_t$ and D_t satisfying $D_t = d_t < \frac{K}{t\beta_l}$, we have

$$\begin{aligned}
&\beta_l D_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, D_t + X_t)] - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&= \beta_l d_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, d_t + X_t)] - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&\leq \beta_l d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, d_t + X_t)] + \beta_c d_t - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&\leq t\beta_l d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)] - (K + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&< 0.
\end{aligned}$$

The first inequality follows from $\sum_{i=1}^N (b_{i,t} + X_{i,t} - X_{i,t})^+ = d_t$ and Lemma 3.3 part (a). The second inequality follows from Lemma 3.3 part (b). Therefore, (18) doesn't hold and it is optimal not to clear.

Next, we consider part two. We only prove the case for $D_t = d_t$, and the claim follows from part four of the proposition directly. For any $\boldsymbol{b}_t$ satisfying $d_t \geq \frac{K}{\beta_l - 2\beta_c}$, we have

$$\begin{aligned}
&\beta_l D_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, D_t + X_t)] - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&= \beta_l d_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, d_t + X_t)] - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&\geq \beta_l d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, d_t + X_t)] - \beta_c d_t - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&\geq (\beta_l - 2\beta_c) d_t - K \\
&\geq 0.
\end{aligned}$$

The first inequality follows from $\sum_{i=1}^N (X_{i,t} - b_{i,t} - X_{i,t})^+ = d_t$ and Lemma 3.3 part (a). The second inequality holds because $V_t(\boldsymbol{b}_t, D_t)$ is increasing in D_t for any t. Therefore, (18) holds and it is optimal to clear.

To show part three, consider $\boldsymbol{b}_t$ and $\boldsymbol{b}'_t$ satisfying $|b_{i,t}| \leq |b'_{i,t}|$ and $b'_{i,t} b_{i,t} \geq 0$. If it is optimal to clear all debts (i.e. the inequality (18) holds) for $(\boldsymbol{b}_t, D_t)$, we are going to show it is optimal to clear for the state $(\boldsymbol{b}'_t, D_t + d'_t - d_t)$. Note that

$$\begin{aligned}
&\beta_l (D_t + d'_t - d_t) + \mathbb{E}[V_{t-1}(\boldsymbol{b}'_t + \boldsymbol{X}_t, D_t + d'_t - d_t + X_t)] - (K + \beta_c d'_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&\geq \beta_l D_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, D_t + d'_t - d_t + X_t)] + (\beta_l - \beta_c)(d'_t - d_t) - (K + \beta_c d'_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&\geq \beta_l D_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, D_t + X_t)] - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) + (\beta_l - 2\beta_c)(d'_t - d_t) \\
&\geq \beta_l D_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, D_t + X_t)] - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
&\geq 0.
\end{aligned}$$

The first inequality follows from $\sum_{i=1}^N (b_{i,t} + X_{i,t} - (b'_{i,t} + X_{i,t}))^+ = d'_t - d_t \geq 0$ (because $|b_{i,t}| \leq |b'_{i,t}|$ and $b'_{i,t} b_{i,t} \geq 0$) and Lemma 3.3 part (a). The second inequality holds because $V_t(\boldsymbol{b}_t, D_t)$ is increasing in D_t and $d'_t \geq d_t$. The third inequality follows from the condition that $\beta_l \geq 2\beta_c$. The last inequality is due to the fact that (18) holds for $(\boldsymbol{b}_t, D_t)$. Therefore, (18) holds for $(\boldsymbol{b}'_t, D_t + d'_t - d_t)$ and it is optimal to clear.

To show part four, note that the existence of such a threshold $g_t(\boldsymbol{b}_t$ can be obtained from (18): the LHS is increasing in D_t and the RHS is independent of D_t. To obtain the range of the threshold, note that $D_t \geq d_t = \sum_{i=1}^N b_{i,t}^+$, which gives the lower bound of $g_t(\cdot)$. To obtain the upper bound, it suffices to show that for $(\boldsymbol{b}_t, D_t)$ satisfying $D_t \geq \frac{K}{\beta_l} + \frac{2\beta_c}{\beta_l} d_t$, it is optimal to clear. Note that (18) can be written as

$$\begin{aligned}
&\beta_l D_t + \mathbb{E}[V_{t-1}(\boldsymbol{b}_t + \boldsymbol{X}_t, D_t + X_t)] - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
\geq\, &\beta_l D_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, D_t + X_t)] - \beta_c d_t - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(\boldsymbol{X}_t, X_t)]) \\
\geq\, &\beta_l D_t - 2\beta_c d_t - K \\
\geq\, &0.
\end{aligned}$$

The first inequality follows from $\sum_{i=1}^N (b_{i,t} + X_{i,t} - X_{i,t})^+ = d_t$ and Lemma 3.3 part (a). The second inequality holds because $V_t(\boldsymbol{b}_t, D_t)$ is increasing in D_t for any t. Thus, it is optimal to clear if $D_t \geq \frac{K}{\beta_l} + \frac{2\beta_c}{\beta_l} d_t$. This completes the proof. □

Proof of Corollary 3.6: We first show that it is optimal to clear if and only if $D_t \geq g_t(|b_{1,t}|)$ for some function $g_t(\cdot)$ satisfying $g_t(x) \geq x$. Suppose it is optimal to clear in period t. By Proposition 3.4, we have $\tilde{D}_t^* = \tilde{d}_t^*$. Based on the Bellman equation (16), it is optimal to clear if and only if $U_t(b_{1,t}, D_t | b_{1,t}, D_t) \geq K + U_t(\tilde{b}_{1,t}^*, \tilde{D}_t^* | b_{1,t}, D_t)$, i.e.,

$$\beta_l D_t + \mathbb{E}[V_{t-1}(b_{1,t} + X_{1,t}, D_t + X_t)] \geq K + \beta_l \tilde{D}_t^* + \beta_c (d_t - \tilde{D}_t^*) + \mathbb{E}[V_{t-1}(\tilde{b}_{1,t}^* + X_{1,t}, \tilde{D}_t^* + X_t)]. \quad (19)$$

By Lemma 3.3 part (a) (left-hand side increasing in D_t), we can uniquely define $D_t = g_t(|b_{1,t}|)$ to be the solution to (19) when the inequality is tight. Moreover, as $D_t \geq d_t = |b_{1,t}|$, we have $g_t(|b_{1,t}|) = D_t \geq |b_{1,t}|$. As we vary $|b_{1,t}|$, this proves part (a).

We next establish part (b). The proof consists of two steps. In the first step, we show that when $\beta_l > 2\beta_c$, the fixed point of $g_t(\cdot)$, i.e. $d_t^c = |b_{1,t}^c|$, is unique, and it is optimal to clear if $D_t = |b_{1,t}| \geq d_t^c$ and not to clear if $D_t = |b_{1,t}| < d_t^c$. By Proposition 3.5, we have $\tilde{b}_{1,t}^* = \tilde{D}_t^* = 0$. Suppose (19) achieves equality for both $D_t = |b_{1,t}^c| = d_t^c$ and $D_t = |b_{1,t}^{c'}| = d_t^{c'}$ with $d_t^{c'} > d_t^c$. We have a contradiction because

$$
\begin{aligned}
0 &= \beta_l d_t^{c'} + \mathbb{E}[V_{t-1}(b_{1,t}^{c'} + X_{1,t}, d_t^{c'} + X_t)] - (K + \beta_c d_t^{c'} + \mathbb{E}[V_{t-1}(X_{1,t}, X_t)]) \\
&\geq \beta_l d_t^{c'} + \mathbb{E}[V_{t-1}(b_{1,t}^{c} + X_{1,t}, d_t^{c} + X_t)] - \beta_c(d_t^{c'} - d_t^c) - (K + \beta_c d_t^{c'} + \mathbb{E}[V_{t-1}(X_{1,t}, X_t)]) \\
&= \beta_l d_t^{c} + \mathbb{E}[V_{t-1}(b_{1,t}^{c} + X_{1,t}, d_t^{c} + X_t)] + (\beta_l - 2\beta_c)(d_t^{c'} - d_t^c) - (K + \beta_c d_t^{c} + \mathbb{E}[V_{t-1}(X_{1,t}, X_t)]) \\
&= (\beta_l - 2\beta_c)(d_t^{c'} - d_t^c) > 0
\end{aligned}
$$

where the first inequality follows from Lemma 3.3 part (a) and Lemma 3.3 part (b) and the last inequality holds because $\beta_l > 2\beta_c$ and $d_t^{c'} > d_t^c$. Thus, the fixed point d_t^c is unique. For any $D_t = |b_{1,t}| \geq d_t^c$, inequality (19) holds and vice versa. This completes the first step.

In the second step, we show that it is optimal to clear if and only if $D_t \geq g_t(|b_{1,t}|)$ for an increasing function $g_t(\cdot)$. Moreover, if $|b_{1,t}| \geq d_t^c$, $g_t(|b_{1,t}|) = |b_{1,t}|$; if $|b_{1,t}| < d_t^c$, $g_t(|b_{1,t}|) > |b_{1,t}|$. Based on the results in the first step, the inequality (19) holds for $D_t = |b_{1,t}| \geq d_t^c$. Then, the inequality (19) holds for all $D_t \geq |b_{1,t}| \geq d_t^c$ because V_{t-1} is increasing in $\tilde{D}_t + X_t$ (Lemma 3.3 part (b)). Obviously, $g_t(|b_{1,t}|) = |b_{1,t}|$ increases with $|b_{1,t}|$.

Similarly, we have from the results in the first step that the inequality (19) does not hold for $D_t = |b_{1,t}| < d_t^c$. Then, there exists a unique $D_t = g_t(|b_{1,t}|) > |b_{1,t}|$ such that (19) achieves equality because V_{t-1} is increasing in $\tilde{D}_t + X_t$ (Lemma 3.3 part (b)) and $\beta_l > 0$. For any $D_t \geq g_t(|b_{1,t}|)$, the inequality (19) holds. We show that the function $g_t(|b_{1,t}|)$ increases with $|b_{1,t}|$. Consider any two $b_{1,t}, b'_{1,t}$ with $|b_{1,t}| < |b'_{1,t}| < d_t^c$. Suppose (19) achieves equality at $(b_{1,t}, g_t(|b_{1,t}|))$ and $(b'_{1,t}, g_t(|b'_{1,t}|))$. Then, we must have $g_t(|b'_{1,t}|) \geq g_t(|b_{1,t}|)$. Otherwise, we have a contradiction that

$$
\begin{aligned}
0 &= \beta_l g_t(d'_t) + \mathbb{E}[V_{t-1}(b'_{1,t} + X_{1,t}, g_t(d'_t) + X_t)] - (K + \beta_c d'_t + \mathbb{E}[V_{t-1}(X_{1,t}, X_t)]) \\
&\leq \beta_l g_t(d'_t) + \mathbb{E}[V_{t-1}(b_{1,t} + X_{1,t}, g_t(d_t) + X_t)] + \beta_c(d'_t - d_t) - (K + \beta_c d'_t + \mathbb{E}[V_{t-1}(X_{1,t}, X_t)]) \\
&= \beta_l g_t(d_t) + \mathbb{E}[V_{t-1}(b_{1,t} + X_{1,t}, g_t(d_t) + X_t)] + \beta_l(g_t(d'_t) - g_t(d_t)) - (K + \beta_c d_t + \mathbb{E}[V_{t-1}(X_{1,t}, X_t)]) \\
&= \beta_l(g_t(d'_t) - g_t(d_t)) < 0
\end{aligned}
$$

where the first inequality follows from Lemma 3.3 part (a) and Lemma 3.3 part (b). This completes the proof of part (b). □

Proof of Proposition 3.7: We first prove that under the condition $\beta_l \geq \beta_c$, if it is optimal to clear, all debts should be cleared. It consists of two steps. In the first step, by induction we prove a variant of the claim: for $t = 0, \cdots, T$, $V_t(\boldsymbol{b}, D) \geq V_t(\boldsymbol{b}', D)$ for $\sum_{i=1}^N (b_i)^+ \geq \sum_{i=1}^N (b'_i)^+$. To use the information for the deterministic process, i.e., x_i, y and z introduced before the proposition, we transform the claim as follows. Let $u_i = b_i/x_i$, $u'_i = b'_i/x_i$, $u = \sum_{i=1}^N (b_i)^+/y$, $u' = \sum_{i=1}^N (b'_i)^+/y$ and $w = D/z$. (If $x_i = 0$, then we can ignore the state b_i because it remains at zero; if $y = 0$, then the problem becomes trivial.) We will show for $t = 0, \cdots, T$, $V_t((u_i x_i), wz) \geq V_t((u'_i x_i), wz)$ for $u_i, u'_i \geq 0$ for all i and $w, u, u' \geq 0$ such that $wz \geq uy = \sum_{i=1}^N (u_i x_i)^+ \geq \sum_{i=1}^N (u'_i x_i)^+ = u'y$.

For $t = 0$, the claim holds because $V_0 \equiv 0$. Suppose the claim holds for period $t-1$. We prove $V_t((u_i x_i), wz) \geq V_t((u'_i x_i), wz)$ in the following four cases: (i) it is optimal to clear for both state $((u_i x_i), wz)$ and $((u'_i x_i), wz)$, (ii) it is optimal to clear for state $((u_i x_i), wz)$ and not to clear for state $((u'_i x_i), wz)$, (iii) it is optimal to clear for state $((u'_i x_i), wz)$ and not to clear for state $((u_i x_i), wz)$, and (iv) it is optimal to not clear for both states.

In the first case, let $((\tilde{u}^*_i x_i), \tilde{w}^* z)$ and $((\tilde{u}'_i x_i), \tilde{w}' z)$ be the optimal solution to $V_t((u_i x_i), wz)$ and $V_t((u'_i x_i), wz)$, respectively. By proposition 3.4, we have $\tilde{w}^* z = \tilde{u}^* y \leq uy$ and $\tilde{w}' z = \tilde{u}' y \leq u'y$. If $\tilde{u}^* y < \tilde{u}' y$, we can choose $\tilde{u}''_i \in [0, u'_i], \forall i$ such that $((\tilde{u}''_i x_i), \tilde{u}^* y)$ is a feasible solution to $V_t((u'_i x_i), wz)$ where $\sum_{i=1}^N (\tilde{u}''_i x_i)^+ = \sum_{i=1}^N (\tilde{u}^*_i x_i)^+ = \tilde{u}^* y$. Then, we have a contradiction because

$$
\begin{aligned}
V_t((u'_i x_i), wz) &= K + \beta_l \tilde{u}' y + \beta_c (u'y - \tilde{u}' y) + V_{t-1}((\tilde{u}'_i x_i + x_i), \tilde{u}' y + z) \\
&> K + \beta_l \tilde{u}' y + \beta_c (u'y - \tilde{u}' y) + V_{t-1}((\tilde{u}''_i x_i + x_i), \tilde{u}' y + z) \\
&> K + \beta_l \tilde{u}^* y + \beta_c (u'y - \tilde{u}^* y) + V_{t-1}((\tilde{u}''_i x_i + x_i), \tilde{u}^* y + z) + (\beta_l - \beta_c)(\tilde{u}' y - \tilde{u}^* y) \\
&> K + \beta_l \tilde{u}^* y + \beta_c (u'y - \tilde{u}^* y) + V_{t-1}((\tilde{u}''_i x_i + x_i), \tilde{u}^* y + z)
\end{aligned}
$$

where the first inequality follows from the induction hypothesis, the second inequality follows from Lemma 3.3 part (b), and the last inequality holds because $\beta_l \geq \beta_c$. Thus, $\tilde{u}^* y \geq \tilde{u}' y$ and

$$
\begin{aligned}
V_t((u_i x_i), wz) - V_t((u'_i x_i), wz) &= K + \beta_l \tilde{u}^* y + \beta_c (uy - \tilde{u}^* y) + V_{t-1}((\tilde{u}^*_i x_i + x_i), \tilde{u}^* y + z) \\
&\quad - \big(K + \beta_l \tilde{u}' y + \beta_c (u'y - \tilde{u}' y) + V_{t-1}((\tilde{u}'_i x_i + x_i), \tilde{u}' y + z)\big) \\
&\geq (\beta_l - \beta_c)(\tilde{u}^* y - \tilde{u}' y) + \beta_c (uy - u'y) \qquad (20) \\
&\quad + V_{t-1}((\tilde{u}^*_i x_i + x_i), \tilde{u}^* y + z) - V_{t-1}((\tilde{u}'_i x_i + x_i), \tilde{u}^* y + z) \\
&\geq 0.
\end{aligned}
$$

The first inequality follows from Lemma 3.3 part (b). The second inequality follows from the induction hypothesis and that $\beta_l \geq \beta_c$.

In the second case, let $((\tilde{u}^*_i x_i), \tilde{w}^* z)$ be the optimal solution to $V_t((u_i x_i), wz)$. By proposition 3.4, we have $\tilde{w}^* z = \tilde{u}^* y \leq uy$. Consider $((\tilde{u}'_i x_i), \tilde{u}' y)$ where $\tilde{u}'_i \leq u'_i$ and $\tilde{u}' y = \sum_{i=1}^N (\tilde{u}'_i x_i)^+ \leq \tilde{u}^* y$.

It is a feasible solution to $V_t((u_i'x_i), wz)$. Then,

$$\begin{aligned}
V_t((u_ix_i), wz) - V_t((u_i'x_i), wz) &= K + \beta_l \tilde{u}^* y + \beta_c(uy - \tilde{u}^* y) + V_{t-1}((\tilde{u}_i^* x_i + x_i), \tilde{u}^* y + z) \\
&\quad - \big(\beta_l wz + V_{t-1}((u_i' x_i + x_i), wz + z)\big) \\
&\geq K + \beta_l \tilde{u}^* y + \beta_c(uy - \tilde{u}^* y) + V_{t-1}((\tilde{u}_i^* x_i + x_i), \tilde{u}^* y + z) \\
&\quad - \big(K + \beta_l \tilde{u}' y + \beta_c(u'y - \tilde{u}' y) + V_{t-1}((\tilde{u}_i' x_i + x_i), \tilde{u}' y + z)\big) \\
&\geq 0.
\end{aligned}$$

The first inequality holds because $((u_i'x_i), wz)$ is the optimal policy to $V_t((u_i'x_i), wz)$. The second inequality follows from (20).

In the last two cases, we have

$$\begin{aligned}
V_t((u_ix_i), wz) - V_t((u_i'x_i), wz) &\geq \beta_l wz + V_{t-1}((u_i x_i + x_i), wz + z) \\
&\quad - \big(\beta_l wz + V_{t-1}((u_i' x_i + x_i), wz + z)\big) \\
&\geq 0.
\end{aligned}$$

The last inequality follows from the induction hypothesis. Hence we have completed the first step.

In the second step, we show that if it is optimal to clear, all debts should be cleared, i.e. $\tilde{u}_i^* = 0, \forall i$ and $\tilde{w}^* = 0$, following the same notations as in the first step. We will prove this claim by contradiction. Suppose $\tilde{w}^* > 0$. By Proposition 3.4, we also have $\tilde{w}^* z = \tilde{u}^* y > 0$. By the definition (17) of the function U, we have

$$\begin{aligned}
U_t\left((\tilde{u}_i^* x_i), \tilde{u}^* y | (u_i x_i), wz\right) &= \beta_c(uy - \tilde{u}^* y) + \beta_l \tilde{u}^* y + V_{t-1}((\tilde{u}_i^* x_i + x_i), \tilde{u}^* y + z) \\
&\geq \beta_c(uy - \tilde{u}^* y) + \beta_l \tilde{u}^* y + V_{t-1}((\tilde{u}_i^* x_i + x_i), z) \\
&\geq \beta_c uy + (\beta_l - \beta_c)\tilde{u}^* y + V_{t-1}((x_i), z) \\
&\geq \beta_c uy + V_{t-1}((x_i), z) \\
&= U_t(\mathbf{0}, 0 | (u_i x_i), wz).
\end{aligned}$$

The first inequality follows from Lemma 3.3 part (b). The second inequality follows from step one. The third inequality follows from the condition that $\beta_l \geq \beta_c$. By the Bellman equation (16), it is optimal to use $(\mathbf{0}, 0)$ instead of $((\tilde{u}_i^* x_i), \tilde{u}^* y)$, which leads to a contradiction. This completes the proof of the clear-all claim.

Next, we prove that the clearing cycles only have two values of length in two steps. In the first step, we show that *given* the total number of clearings c, it is optimal to clear every $\tau = \left\lfloor \frac{T}{c} \right\rfloor$ periods for the first $c(\tau + 1) - T$ times and every $\tau + 1$ periods for the last $T - c\tau$ times.

When $c=1$, all debts are cleared only at period $t=0$, i.e., $\tau=T$. (Recall that we always clear the debt at $t=0$ at no cost, i.e., $V_0 \equiv 0$.) Suppose the claim holds for $c-1$ times of clearing. We prove the claim for c by induction. Suppose the first clearing occurs in period $t \in [c-1, T)$. The clearing cost of the first clearing is

$$\begin{aligned} \mathcal{G}_0(t) = &\ \beta_l z + 2\beta_l z + \cdots + (T-t-1)\beta_l z + K + (T-t)\beta_c y \\ = &\ \frac{(T-t)(T-t-1)}{2}\beta_l z + K + (T-t)\beta_c y. \end{aligned}$$

By the induction hypothesis, let $\tau_1 = \left\lfloor \frac{t}{c-1} \right\rfloor$ and $\tau_1 + 1$ be the length of the clearing cycles in the last t periods ($[0, t-1]$). Let $r_1 = t - (c-1)\tau_1$ be the remainder. Based on the induction hypothesis, the total cost between $[0, t-1]$ is

$$\begin{aligned} \mathcal{G}_1(t) = &\ (c-1-r_1) \times (\beta_l z + 2\beta_l z + \cdots + (\tau_1 - 1)\beta_l z + K + \tau_1 \beta_c y) \\ &+ r_1 \times (\beta_l z + 2\beta_l z + \cdots + \tau_1 \beta_l z) + (r_1 - 1) \times (K + (\tau_1 + 1)\beta_c y) \\ = &\ \frac{(c-1)\tau_1(\tau_1 - 1)}{2}\beta_l z + r_1 \tau_1 \beta_l z + (c-2)K + (t - \tau_1 - 1)\beta_c y \\ = &\ \frac{\tau_1 t - t + r_1 \tau_1 + r_1}{2}\beta_l z + (c-2)K + (t - \tau_1 - 1)\beta_c y. \end{aligned}$$

Then, we solve the optimal length of the first clearing from $t^* = \arg\min\{\mathcal{G}_0(t) + \mathcal{G}_1(t)\}$:

$$\begin{aligned} \mathcal{G}(t) &= \mathcal{G}_0(t) + \mathcal{G}_1(t) \\ &= \frac{(T-t)(T-t-1)}{2}\beta_l z + \frac{\tau_1 t - t + r_1 \tau_1 + r_1}{2}\beta_l z + (c-1)K + (T - \tau_1 - 1)\beta_c y \\ &= \frac{T^2 + t^2 - 2Tt - T + \tau_1 t + r_1 \tau_1 + r_1}{2}\beta_l z + (c-1)K + (T - \tau_1 - 1)\beta_c y \\ &= \frac{\beta_l z}{2}\left(T^2 + t^2 - 2Tt - T + \frac{(t-r_1)t}{c-1} + \frac{r_1(t-r_1)}{c-1} + r_1\right) + (c-1)K + \left(T - \frac{t-r_1}{c-1} - 1\right)\beta_c y \\ &= \frac{c\beta_l z}{2(c-1)}t^2 - \left(T\beta_l z + \frac{\beta_c y}{c-1}\right)t + \frac{\beta_l z}{2}\left(T^2 - T - \frac{r_1^2}{c-1} + r_1\right) + (c-1)K + \left(T + \frac{r_1}{c-1} - 1\right)\beta_c y. \end{aligned}$$

Regardless of the value of r_1, $\mathcal{G}(t)$ achieves the minimum when $t = \frac{(c-1)T\beta_l z + \beta_c y}{c\beta_l z}$, i.e., $T - t = \frac{T}{c} - \frac{\beta_c y}{c\beta_l z}$, if t is allowed to be fractional. Since t has to be an integer, we need to compare the two integers around the fractional value. Let $r = T - c\lfloor T/c \rfloor \triangleq T - c\tau$ where $r \in [0, c)$. We compare the total cost when $t = T - \left\lfloor \frac{T}{c} - \frac{\beta_c y}{c\beta_l z} \right\rfloor$ with that when $t = T - \left\lceil \frac{T}{c} - \frac{\beta_c y}{c\beta_l z} \right\rceil$ in two cases: $r \geq 1$ and $r = 0$.

By definition, we have $z > y > 0$. As $\beta_l \geq \beta_c > 0$, we have $\beta_l z > \beta_c y > 0$. If $r \geq 1$, we have $0 < r - \frac{\beta_c y}{\beta_l z} < c$ and $0 < r < c$. Then, $\lceil \frac{T}{c} - \frac{\beta_c y}{c\beta_l z} \rceil = \lceil \tau + \frac{r}{c} - \frac{\beta_c y}{c\beta_l z} \rceil = \tau + 1$, $\tau_1 = \lfloor \frac{T-\tau-1}{c-1} \rfloor = \lfloor \tau + \frac{r-1}{c-1} \rfloor = \tau$

and $r_1 = c\tau + r - \tau - 1 - (c-1)\tau = r - 1$. In this case, the total cost of $t = T - \left\lceil \frac{T}{c} - \frac{\beta_c y}{c\beta_l z} \right\rceil$ is

$$\mathcal{G}(T-\tau-1) = \frac{c\beta_l z(T-\tau-1)^2}{2(c-1)} - \left(T\beta_l z + \frac{\beta_c y}{c-1}\right)(T-\tau-1) + \frac{\beta_l z}{2}\left(T^2 - T - \frac{(r-1)^2}{c-1} + r - 1\right) + (c-1)K + \left(T + \frac{r-1}{c-1} - 1\right)\beta_c y.$$

Next we investigate the cost for $t = T - \lfloor \frac{T}{c} - \frac{\beta_c y}{c\beta_l z} \rfloor = T - \lfloor \tau + \frac{r}{c} - \frac{\beta_c y}{c\beta_l z} \rfloor = T - \tau$. The total cost $\mathcal{G}(T-\tau)$ is different when $r \in [1, c-2)$ and when $r \in [c-2, c)$. If $r \in [1, c-2)$, we have $\tau_1 = \lfloor \frac{T-\tau}{c-1} \rfloor = \lfloor \tau + \frac{r+1}{c-1} \rfloor = \tau$ and $r_1 = T - \tau - (c-1)\tau = r$. The total cost is

$$\mathcal{G}(T-\tau) = \frac{c\beta_l z(T-\tau)^2}{2(c-1)} - \left(T\beta_l z + \frac{\beta_c y}{c-1}\right)(T-\tau) + \frac{\beta_l z}{2}\left(T^2 - T - \frac{r^2}{c-1} + r\right) + (c-1)K + \left(T + \frac{r}{c-1} - 1\right)\beta_c y.$$

Then, the difference in the total costs is

$$\begin{aligned}
\mathcal{G}(T-\tau) - \mathcal{G}(T-\tau-1) &= \frac{c\beta_l z(2T-2\tau-1)}{2(c-1)} - \left(T\beta_l z + \frac{\beta_c y}{c-1}\right) + \frac{\beta_l z}{2}\left(1 - \frac{2r-1}{c-1}\right) + \frac{\beta_c y}{c-1} \\
&= c\tau\beta_l z + \frac{r\beta_l z}{2} - \frac{r\beta_l z}{2(c-1)} - T\beta_l z \\
&= -\frac{r\beta_l z}{2} - \frac{r\beta_l z}{2(c-1)} < 0.
\end{aligned}$$

This implies that $t^* = T - \tau$ when $1 \leq r < c-2$.

If $r \in [c-2, c)$, we have $\tau_1 = \tau + 1$ and $r_1 = T - \tau - (c-1)(\tau+1) = r - (c-1)$. As $r_1 \in [0, c-1)$, we have $r = c-1$ and $r_1 = 0$. The total cost is

$$\mathcal{G}(T-\tau) = \frac{c\beta_l z(T-\tau)^2}{2(c-1)} - \left(T\beta_l z + \frac{\beta_c y}{c-1}\right)(T-\tau) + \frac{\beta_l z}{2}\left(T^2 - T\right) + (c-1)K + (T-1)\beta_c y.$$

The difference in the total costs is

$$\begin{aligned}
\mathcal{G}(T-\tau) - \mathcal{G}(T-\tau-1) &= \frac{c\beta_l z(2T-2\tau-1)}{2(c-1)} - \left(T\beta_l z + \frac{\beta_c y}{c-1}\right) + \frac{\beta_l z}{2}\left(\frac{(r-1)^2}{c-1} - r + 1\right) + \frac{r-1}{c-1}\beta_c y \\
&= c\tau\beta_l z + \frac{c-1}{2}\beta_l z - \frac{c-3}{c-1}\beta_c y - T\beta_l z \\
&= -\frac{r\beta_l z}{2} - \frac{r-2}{r}\beta_c y \\
&\leq 0.
\end{aligned}$$

This implies that $t^* = T - \tau$ when $r = c-1$.

Thus, we have $T - t^* = \tau$ if $r \geq 1$. Moreover, $\tau_1 = \tau$ and $r_1 = r$ if $r \in [1, c-2)$, and $\tau_1 = \tau + 1$ if $r = c-1$. That is, it is optimal to clear all the debts every τ periods for the first $c(\tau+1) - T$ times and every $\tau + 1$ periods for the last $T - c\tau$ times.

If $r=0$, as $0<\frac{\beta_c y}{\beta_l z}<1$, we have $\lceil \frac{T}{c}-\frac{\beta_c y}{c\beta_l z}\rceil=\lceil \tau+\frac{r}{c}-\frac{\beta_c y}{c\beta_l z}\rceil=\tau$, $\tau_1=\lfloor\frac{T-\tau}{c-1}\rfloor=\tau$ and $r_1=c\tau-\tau-(c-1)\tau=0$. The total cost for $t=T-\lceil \tau+\frac{r}{c}-\frac{\beta_c y}{c\beta_l z}\rceil=T-\tau$ is

$$\mathcal{G}(T-\tau)=\frac{c\beta_l z(T-\tau)^2}{2(c-1)}-\left(T\beta_l z+\frac{\beta_c y}{c-1}\right)(T-\tau)+\frac{\beta_l z}{2}(T^2-T)+(c-1)K+(T-1)\beta_c y.$$

Moreover, $\lfloor \frac{T}{c}-\frac{\beta_c y}{c\beta_l z}\rfloor=\tau-1$. The total cost for $t=T-\lfloor \tau+\frac{r}{c}-\frac{\beta_c y}{c\beta_l z}\rfloor=T-\tau+1$ is different when $c=2$ and $c>2$. When $c=2$, we have $\tau_1=\lfloor\frac{c\tau-\tau+1}{c-1}\rfloor=\lfloor\tau+\frac{1}{c-1}\rfloor=\tau+1$ and $r_1=c\tau-\tau+1-(c-1)(\tau+1)=0$. The total cost is

$$\mathcal{G}(T-\tau+1)=\beta_l z(T-\tau+1)^2-(T\beta_l z+\beta_c y)(T-\tau+1)+\frac{\beta_l z}{2}\left(T^2-T\right)+K+(T-1)\,\beta_c y.$$

The difference in the total costs is

$$\mathcal{G}(T-\tau+1)-\mathcal{G}(T-\tau)=(2T-2\tau+1)\beta_l z-(T\beta_l z+\beta_c y)=\ \beta_l z-\beta_c y>\ 0.$$

This implies that $t^*=T-\tau$ when $r=0$ and $c=2$.

When $c>2$, we have $\tau_1=\lfloor\tau+\frac{1}{c-1}\rfloor=\tau$ and $r_1=c\tau-\tau+1-(c-1)\tau=1$. The total cost is

$$\begin{aligned}\mathcal{G}(T-\tau+1)=&\frac{c\beta_l z(T-\tau+1)^2}{2(c-1)}-\left(T\beta_l z+\frac{\beta_c y}{c-1}\right)(T-\tau+1)+\frac{\beta_l z}{2}\left(T^2-T+\frac{c-2}{c-1}\right)\\&+(c-1)K+\left(T-\frac{c-2}{c-1}\right)\beta_c y.\end{aligned}$$

The difference in the total costs is

$$\mathcal{G}(T-\tau+1)-\mathcal{G}(T-\tau)=\frac{c\beta_l z(2T-2\tau+1)}{2(c-1)}-\left(T\beta_l z+\frac{\beta_c y}{c-1}\right)+\frac{\beta_l z}{2}\times\frac{c-2}{c-1}+\frac{1}{c-1}\beta_c y=\ \beta_l z>\ 0.$$

This implies that $t^*=T-\tau$ when $r=0$ and $c>2$. Thus, we have $T-t^*=\tau$ if $r=0$. Moreover, $\tau_1=\tau$ and $r_1=0$. Thus, the claim holds for c times of clearing for an arbitrary c. Under the optimal t^*, it is easy to check that there are only two possible clearing cycles of length τ and $\tau+1$. This completes the first step.

In the second step, we show that the optimal clearing cycle is $\tau^*=\left\lfloor\sqrt{\frac{2K}{\beta_l z}}\right\rfloor$ when $T\to\infty$. We define $\mathcal{F}(\tau)$ to be the limiting clearing cost per period when the clearing cycle length is τ and $T=c\tau$ for an integer c as $c\to\infty$. That is,

$$\begin{aligned}\mathcal{F}(\tau)=&\lim_{c\to\infty}\frac{1}{T}\left(c\times\frac{\tau(\tau-1)}{2}\beta_l z+(c-1)\times(K+\tau\beta_c y)\right)\\=&\lim_{c\to\infty}\left(\frac{\tau-1}{2}\beta_l z+\frac{K}{\tau}+\beta_c y-\frac{K+\tau\beta_c y}{c\tau}\right)\\=&\lim_{c\to\infty}\left(\frac{\tau-1}{2}\beta_l z+\frac{K}{\tau}+\beta_c y\right).\end{aligned}$$

By taking the derivative, it is easy to see that $\mathcal{F}(\tau)$ reaches the minimum at $\tau^* = \left\lfloor \sqrt{\frac{2K}{\beta_l z}} \right\rfloor$.

Now we consider the regime $T \to \infty$. For a given T, by the first step, the cycle length only consists of two values τ and $\tau + 1$. That is, the horizon consists of c_1 cycles of length τ and c_2 cycles of length $\tau + 1$, where $T = T_1 + T_2$, $T_1 = c_1\tau$ and $T_2 = c_2(\tau + 1)$. Therefore, the average clearing cost is

$$\frac{1}{T}\left(T_1\mathcal{F}(\tau) + T_2(\mathcal{F}(\tau+1) + K + \tau\beta_c y\right).$$

Note that $T_1\mathcal{F}(\tau) + T_2(\mathcal{F}(\tau + 1)$ is a convex combination of $\mathcal{F}(\tau)$ and $\mathcal{F}(\tau + 1)$, and is always no less than $\mathcal{F}(\tau^*)$. Meanwhile, we can achieve the average clearing cost $\mathcal{F}(\tau^*)$ as $T \to \infty$ by letting $\tau = \tau^*$, $c_1 = \lfloor T/\tau_1 \rfloor$ and $c_2 = T - c_1\tau$. This completes the proof. □